MATH 200X FALL 2007 FINAL EXAM

December 17, 2007
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NAME: Solutions

There are 8 questions on this exam for a total of 110 points. You may not use calculators, books, or notes. You must show your work to receive full credit. You should simplify numerical answers.

You have two hours to complete the exam.

May the Force be with you.

problem	points
1	
2	
3	
4	
5	
6	
7	
8	
total	

1. (5 points each) Evaluate the limits below. Give the most complete answer. Show your work or explain your reasoning.

(a) $\lim_{x\to 0} \frac{x^4 - 2x^2}{x^3 + x^2}$ $= \lim_{x\to 0} \frac{x^2(x^2-2)}{x^2(x+1)} = \lim_{x\to 0} \frac{x^2-2}{x+1} = \frac{-2}{1} = -2$

(b) $\lim_{x\to -5^+} \frac{e^x}{x^2 - 25}$ $= -\infty$

As $x \to -5^+$, e^x approaches e^{-5}.

something like -4.9, -4.99 or -4.999.

Since x is a little larger than -5, x^2 is a little smaller than 25.

So as $x\to -5^+$, x^2-25 approaches 0 but on the negative side.

(c) $\lim_{x\to 0} \frac{e^{2x} - 1}{3x + x^2}$ $\overset{H}{=} \lim_{x\to 0} \frac{2e^{2x}}{3+2x} = \frac{2e^0}{3+2\cdot 0} = \frac{2}{3}$

form $\frac{0}{0}$

(d) $\lim_{x\to\infty} \frac{5 + \sqrt{2x^4 + x}}{7x^2 + x^{2/3}}$ $\cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{5}{x^2} + \sqrt{2 + \frac{1}{x^3}}}{7 + \frac{1}{x^{4/3}}} = \frac{0 + \sqrt{2+0}}{7+0} = \frac{\sqrt{2}}{7}$

2. (5 points each) Find dy/dx for each of the following.

(a) $y = e^{2x}\cos(x^3 + e^2)$

$$\frac{dy}{dx} = 2e^{2x} \cdot \cos(x^3+e^2) + e^{2x} \cdot (-\sin(x^3+e^2) \cdot 3x^2)$$

$$= e^{2x}\left[2\cos(x^3+e^2) - 3x^2\sin(x^3+e^2)\right]$$

(b) $y = \ln(x + \sec(3x))$

$$\frac{dy}{dx} = \frac{1}{x+\sec(3x)} \cdot (1 + 3\sec(3x)\tan(3x))$$

$$= \frac{1 + 3\sec(3x)\tan(3x)}{x + \sec(3x)}$$

(c) $xe^y = y - 1$

$$1 \cdot e^y + x \cdot e^y \frac{dy}{dx} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{e^y}{1 - xe^y}$$

$$e^y = \frac{dy}{dx} - xe^y \frac{dy}{dx}$$

$$e^y = (1 - xe^y)\frac{dy}{dx}$$

3. (5 points each) Evaluate the following integrals.

(a) $\int \frac{7\sin\theta}{(\cos\theta)^2}d\theta$ $= 7\int (\cos\theta)^{-2}\sin\theta\, d\theta = -7\int u^{-2}\,du$

let $u = \cos\theta$

$du = -\sin\theta\, d\theta$

$-du = \sin\theta\, d\theta$

$= 7u^{-1} + C$

$= 7(\cos\theta)^{-1} + C$

(b) $\int_1^2 \frac{e^{1/t}}{t^2}dt$ $= -\int_1^{1/2} e^u\,du = -e^u\Big]_1^{1/2} = -e^{1/2} + e^1$

$= e - \sqrt{e}$

let $u = \frac{1}{t} = t^{-1}$

$du = -t^{-2}dt$

$-du = t^{-2}dt$

when $t=1$, $u = \frac{1}{1} = 1$

$t=2$, $u = \frac{1}{2}$

(c) $\int_{1/2}^1 \sin(\pi x)\cos(\pi x)dx$ $= \frac{1}{\pi}\int_1^0 u\,du = \frac{1}{\pi}\cdot\frac{1}{2}\cdot u^2\Big]_1^0 = \frac{1}{2\pi}(0^2 - 1^2)$

$= \frac{-1}{2\pi}$

let $u = \sin(\pi x)$

$du = \pi\cos(\pi x)dx$

$\frac{1}{\pi}du = \cos(\pi x)dx$

when $x = \frac{1}{2}$, $u = \sin\frac{\pi}{2} = 1$

$x = 1$, $u = \sin\pi = 0$

4. (5 points each) Let $f(x) = \frac{x^2+x+1}{x^2}$; $f'(x) = -\frac{x+2}{x^3}$; $f''(x) = \frac{2(x+3)}{x^4}$

(a) Identify any asymptotes and show your answer is correct.

horizontal asymptote at $y=1$: $\lim_{x \to \pm\infty} \frac{x^2+x+1}{x^2} = 1$.

vertical asymptote at $x=0$: $\lim_{x \to 0^+} \frac{x^2+x+1}{x^2} = \infty$.

(b) Identify the x-value of any local extrema and show your answer is correct.

Set $f'=0$: $0 = -\left(\frac{x+2}{x^3}\right)$. So $x+2=0$ or $\underline{\underline{x=-2}}$

And, f' is undefined at $\underline{\underline{x=0}}$.

$- - - -\ 0 + + +\ \xi\ - - - -$ ← sign of f'

$-10 \quad -2 \quad -1 \quad 0 \quad 1$

answer

f has a local min at $x=-2$.

Since $x=0$ is not in the domain, there is no extremum at $x=0$.

(c) Identify the x-values of any inflection points and show your answer is correct.

$f''=0$ when $x+3=0$ or $x=-3$

f'' undefined at $x=0$, not in the domain.

$- - -\ 0 + + +\ \xi + + +$ ← sign of f''

$-3 \quad 0$

answer

f has an inflection point at $x=-3$

5. (10 points) Find the equation of the line tangent to the curve $f(x) = \frac{1}{2+2\sin x}$ at $x = 0$.

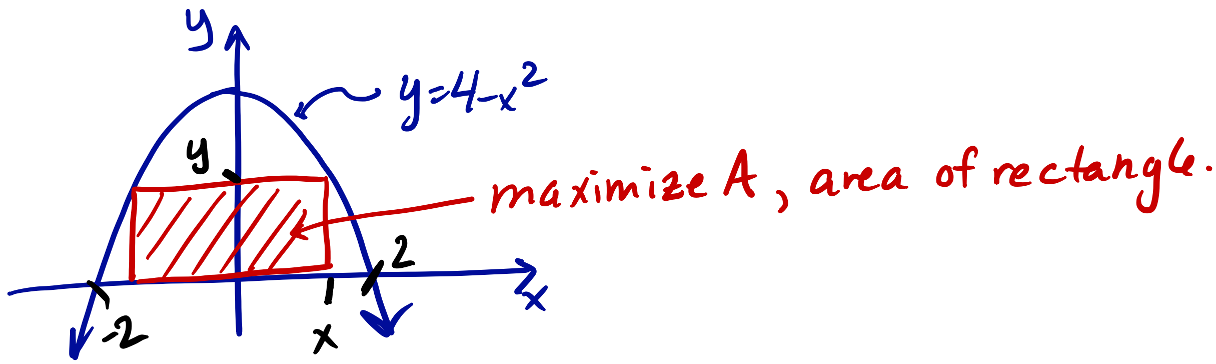
at $x=0$, $f(0) = \dfrac{1}{2+2\sin 0} = \dfrac{1}{2}$. point: $(0, \frac{1}{2})$

$f(x) = (2+2\sin x)^{-1}$. So $f'(x) = -1(2+2\sin x)^{-2} \cdot 2\cos x$.

So $f'(0) = -1(2+2\sin 0)^{-2} \cdot 2\cos 0 = \dfrac{-2}{2^2} = \dfrac{-2}{4} = \dfrac{-1}{2} = m$

line: $y - \frac{1}{2} = \frac{-1}{2}(x-0)$ or $y = \frac{-1}{2}x + \frac{1}{2}$

6. (10 points) Let R be the region below the graph of $y = 4 - x^2$ and above the x-axis. Find the dimensions of the rectangle of largest area that can be inscribed in R assuming the base of the rectangle is on the x-axis.



$A = 2xy = 2x(4-x^2) = 8x - 2x^3$ on $[0,2]$.

$A'(x) = 8 - 6x^2 = 0$ or $x^2 = \frac{8}{6} = \frac{4}{3}$. So $x = \pm 2/\sqrt{3}$.

- Does $x = 2/\sqrt{3}$ correspond to a maximum?

$A''(x) = -12x$, $A''\left(\frac{2}{\sqrt{3}}\right) = -12\left(2/\sqrt{3}\right) < 0$

So A has a max at $x = 2/\sqrt{3}$ from the Second Derivative Test.

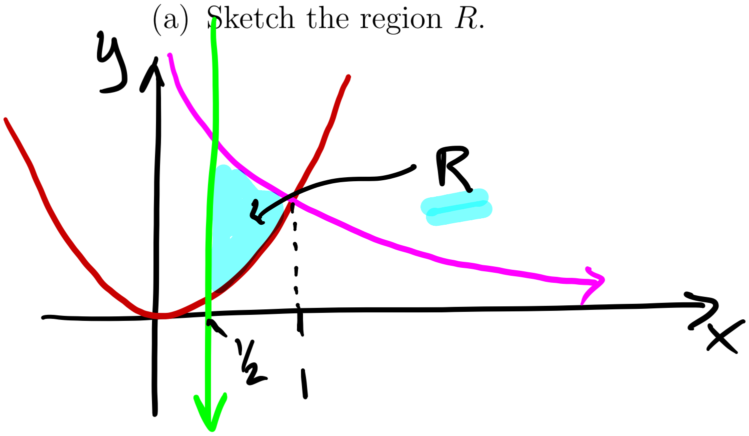
- Dimensions?

If $x = 2/\sqrt{3}$, then the width is $4/\sqrt{3}$ and the height is $4 - \frac{4}{3} = \frac{8}{3}$

* In 2017, this topic does not appear in Calculus I.

~~7.~~ (5 points each) Let R be the region bounded by $y = x^2$, $y = \frac{1}{x}$, and $x = 1/2$.

(a) Sketch the region R.



(b) Set up but do not evaluate an integral for the volume of the solid obtained by rotating the region R about the x-axis.

$$V = \pi \int_{1/2}^{1} \left(\frac{1}{x}\right)^2 - \left(x^2\right)^2 \, dx$$

(c) Set up but do not evaluate an integral for the volume of the solid obtained by rotating the region R about the line $x = 3$.

$$V = 2\pi \int_{1/2}^{1} (3-x)\left(x^2 - \tfrac{1}{2}\right) dx$$

8. (2 points each) Answer the following.

(a) (Complete the following definition.) The derivative of a function $f(x)$ is denoted $f'(x)$ as is defined as:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h},$$

provided...

the limit exists.

(b) (Complete the following statement.) If $\frac{dy}{dx} = 35y$ and $y(0) = 12$, then

$$y = 12e^{35x}$$

(c) Give an example of a definite integral for which the Fundamental Theorem of Calculus does not apply and explain why it does not apply.

$\int_{-1}^{1} \frac{1}{x}\,dx$ $f(x)=\frac{1}{x}$ is not defined on $[-1,1]$

(d) Give an example of a function $f(x)$ and an x-value a such that f is continuous at $x = a$ but not differentiable at $x = a$.

$f(x)=|x|$ is continuous but not differentiable at $x=0$.

(e) Assume $d(x)$ measures the density of a wire, in grams per centimeter (g/cm), where x is measured in centimeters (cm) from one end of the wire. Interpret the statement $d'(3) = -0.5$.

3 cm from one end, the density of the wire is decreasing at a rate of 0.5 g/cm²

ENJOY YOUR HOLIDAY!