# Math F251 Midterm 1 Fall 2018

**Name:** Solutions

**Student Id:** ____________

**Section:**
- ☐ F01 (Bueler)
- ☐ F02 (Jurkowski)
- ☐ F03 (Maxwell)

## Rules:

You have 60 minutes to complete the exam.

Partial credit will be awarded, but you must show your work.

No calculators, books, notes, or other aids are permitted.

Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question where appropriate.

If you need extra space, you can use the back sides of the pages. Please make it obvious when you have done so.

Turn off anything that might go beep during the exam.

Good luck!

| Problem | Possible | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 20 | |
| 3 | 10 | |
| 4 | 25 | |
| 5 | 12 | |
| 6 | 9 | |
| Extra Credit | 3 | |
| Total | 86 | |

**1. (10 points)**

For a particular function $f(x)$,

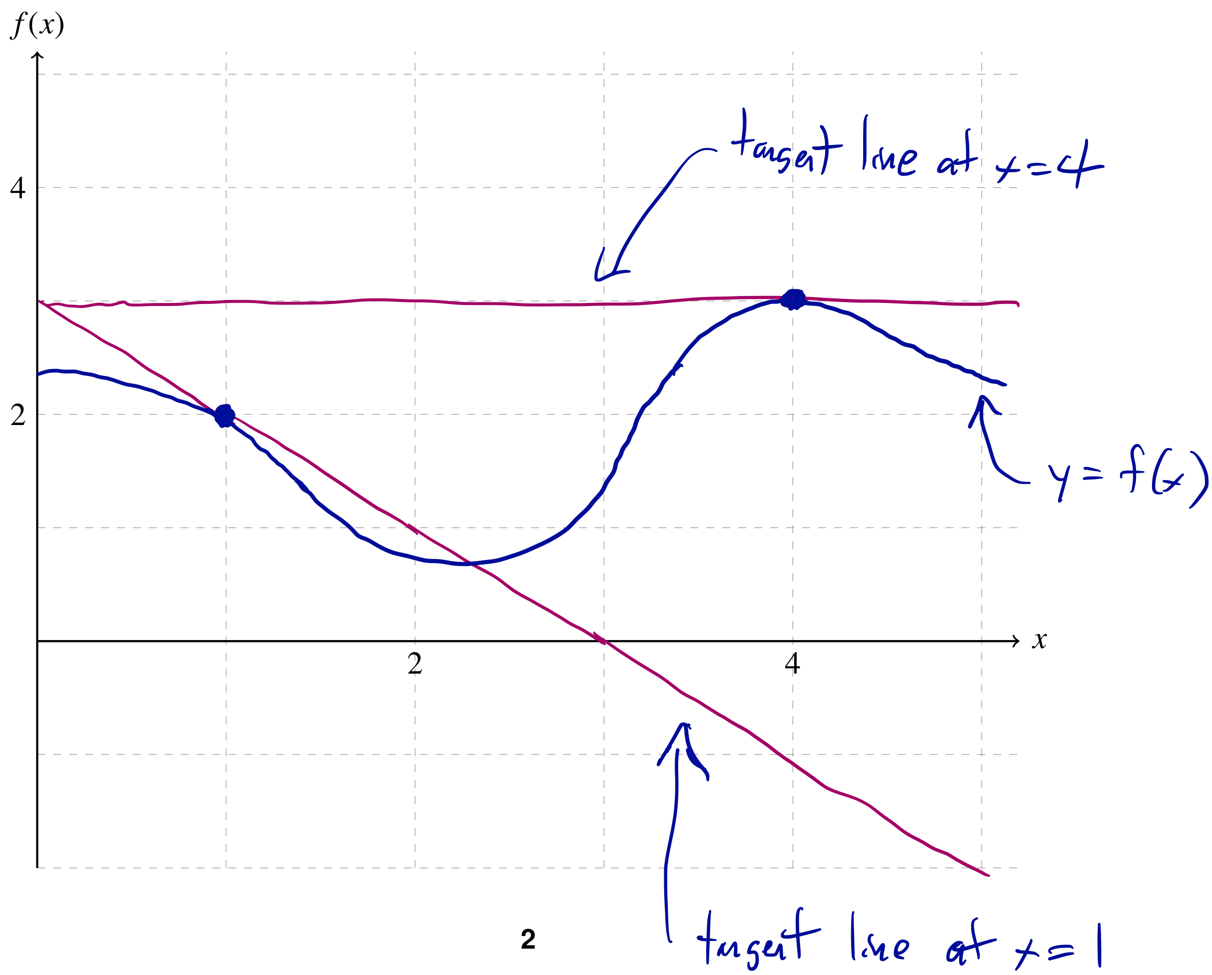
$$f(1) = 2, \quad f(4) = 3, \quad f'(1) = -1 \quad \text{and} \quad f'(4) = 0.$$

**a.** Find the equation of the tangent line of the graph of $f(x)$ at $x = 1$.

$y = 2 - (x-1)$

**b.**

- On the axes below make a rough sketch of what the graph $y = f(x)$ might look like, using all of the limited information that you have.
- Sketch the tangent line at $x = 1$.
- Sketch the tangent line at $x = 4$.



**2. (20 points)**

Compute the following limits, or explain why the given limit does not exist.

**a.** $\lim_{h\to 0} \frac{\sqrt{4+h}-2}{h}$ $= \lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}$

$$= \lim_{h\to 0} \frac{4+h-4}{h(\sqrt{4+h}+2)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2} = \frac{1}{\sqrt{4}+2} = \boxed{\frac{1}{4}}$$

**b.** $\lim_{x\to 5^+} \frac{x^2-3x}{5-x}$ $= \frac{25-15}{0^-} = \frac{10}{0^-} = \boxed{-\infty}$

since $5-x \to 0^-$ as $x \to 0^+$.

**c.** $\lim_{x\to\infty} \frac{\sqrt{x^2-1}}{2-3x}$ $= \lim_{x\to\infty} \frac{x\sqrt{1-1/x^2}}{2-3x}$

$$= \lim_{x\to\infty} \frac{\sqrt{1-1/x^2}}{\frac{2}{x}-3} = \frac{\sqrt{1-0}}{0-3} = \boxed{-\frac{1}{3}}$$

**d.** $\lim_{x\to\infty} x\sin(x)$ $\boxed{\text{does not exist.}}$

As $x\to\infty$, $\sin(x)$ oscillates between $\pm 1$ and $x\sin(x)$ oscillates between arbitrarily large positive and negative numbers.

**3. (10 points)**

Suppose

$$f(x) = \frac{x^2 - 4}{x + 2}.$$

**a.** What is the domain of this function?

$(-\infty, -2) \cup (-2, \infty)$

**b.** State the definition of "The function $f(x)$ is continuous at $x = a$".

$\lim_{x \to a} f(x) = f(a)$

**c.** Determine the choice of value for $f(-2)$ that will make the function $f(x)$ from part **a)** continuous at $x = -2$.

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{x^2 - 4}{x + 2} = \lim_{x \to -2} \frac{(x-2)(x+2)}{x+2}$$

$$= \lim_{x \to -2} x - 2 = -4$$

So pick $f(-2) = \boxed{-4.}$

**4. (25 points)**

For time $t \geq 1$ hours the volume of water in a rain barrel is

$$v(t) = 50 - \frac{20}{t^2}$$

gallons. Answer the following questions about the function $v(t)$ and be sure to include **units** in your answers.

**a.** How much rain is in the barrel at time $t = 2$ hours?

$v(2) = 50 - \frac{20}{4} = 45$ gallons

**b.** Compute $\lim_{t \to \infty} v(t)$. Then explain what this quantity means in precise but everyday language that the general public would understand.

$\lim_{t \to \infty} v(t) = 50 - 20 \cdot 0 = 50$ gallons.

"Eventually, the barrel contains 50 gallons of water."

**Continued....**

**Problem 4 continued....**

Recall that the volume of water in the rain barrel is

$$v(t) = 50 - \frac{20}{t^2}$$

gallons at time $t \geq 1$ hours.

**c.** Compute the average rate of change (with units) of the volume of rain in the barrel from time $t = 1$ to time $t = 2$ hours.

$$\frac{\Delta v}{\Delta t} = \frac{v(2) - v(1)}{2-1} = \frac{\left(50 - \frac{20}{4}\right) - (50 - 20)}{2-1} = \frac{45 - 30}{1}$$

$$= \boxed{15 \text{ gallons/hour}}$$

**d.** Using the limit definition of the derivative, compute $v'(1)$. [**No credit** will be awarded for computing the derivative using any approach other than the limit definition.]

$$v'(1) = \lim_{a \to 1} \frac{v(a) - v(1)}{a-1} = \lim_{a \to 1} \frac{\left(50 - \frac{20}{a^2}\right) - \left(50 - \frac{20}{1}\right)}{a-1}$$

$$= \lim_{a \to 1} \frac{\frac{20a^2 - 20}{a^2}}{a-1}$$

$$= \lim_{a \to 1} \frac{20(a+1)(a-1)}{a^2(a-1)}$$

$$= \lim_{a \to 1} \frac{20(a+1)}{a^2}$$

$$= \frac{20 \cdot 2}{1} = \boxed{40 \text{ gallons/hour}}$$

**e.** What is the instantaneous rate of change (with units) of the volume of water in the barrel at time $t = 1$ hour?

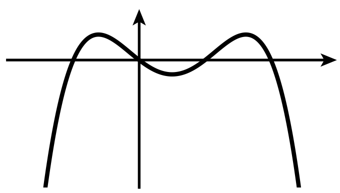
$\boxed{40 \text{ gallons/hour}}$

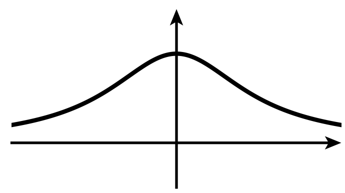
**5. (12 points)**

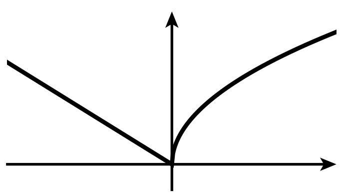
Match the graph of each function (a) - (d) with the graph of its derivative I-VIII. Write your answers in the blanks provided below.

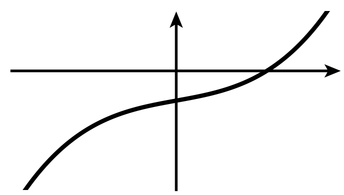
1. The derivative of graph (a) is IV
2. The derivative of graph (c) is II
3. The derivative of graph (b) is III
4. The derivative of graph (d) is IX

**Functions:**

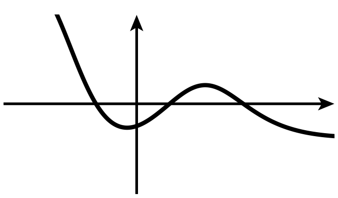
a) 

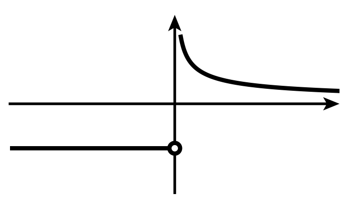
b) 

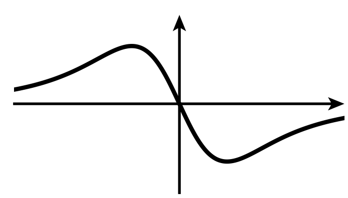
c) 

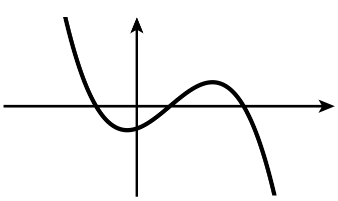
d) 

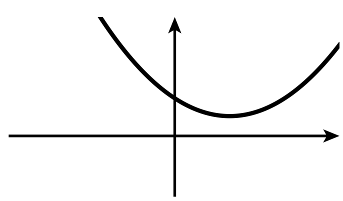
**Derivatives:**

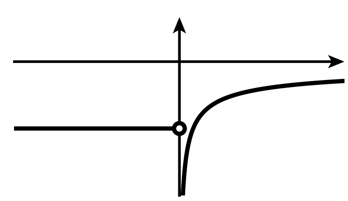
I. 

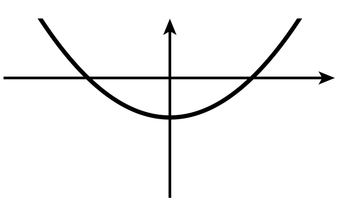
II. 

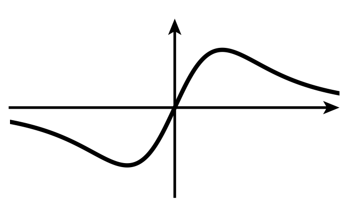
III. 

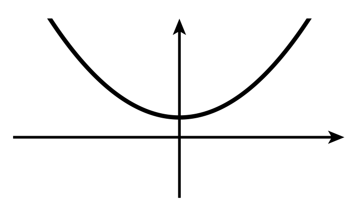
IV. 

V. 

VI. 

VII. 

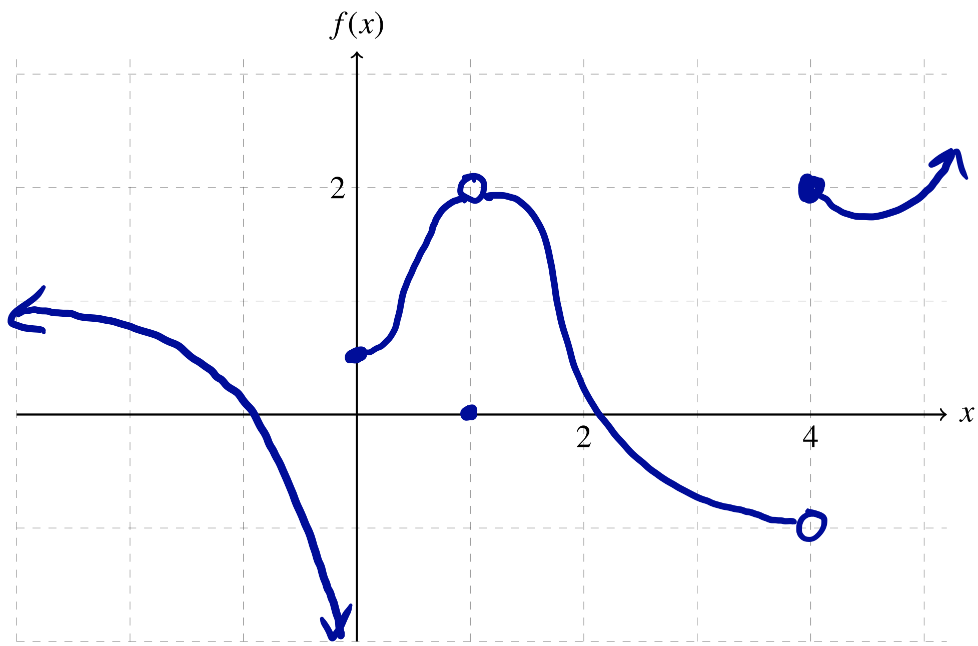
VIII. 

IX. 

**6. (9 points)**

In the diagram below, sketch the graph of a function $f(x)$ satisfying the following criteria.

1. $f(1) = 0$
2. $f(0) = 1/2$
3. $\lim_{x\to 1} f(x) = 2$
4. $\lim_{x\to 0^-} f(x) = -\infty$
5. $\lim_{x\to -\infty} f(x) = 1$
6. $\lim_{x\to 0^+} f(x) = 1/2$
7. $\lim_{x\to 4^-} f(x) = -1$
8. $\lim_{x\to 4^+} f(x) = 2$
9. $f(4) = 2$




**7. (Extra Credit: 3 points)**

Consider the function $f(x) = x^2 - \cos(x)$. Show that there is a value of $x$ such that $f(x) = 1000$.

Observe $f(30) = 900 - \cos(30) \leq 901$.
And $f(32) = 1024 - \cos(32) \geq 1023$.

Since $f(x)$ is continuous on $[30, 32]$, and since $f(30) < 1000 < f(32)$, the Intermediate Value Theorem implies there is an $x$ in $[30, 32]$ such that $f(x) = 1000$.

[n.b. other test choices besides 30, 32 are legitimate. E.g. 0 and 100 would work as well.]