Math F251 **Midterm 2** **Fall 2018**

Name: SOLUTIONS

Student Id: ____________________

Section: ☐ F01 (Bueler) ☐ F02 (Jurkowski) ☐ F03 (Maxwell)

Rules:

You have 60 minutes to complete the exam.

Partial credit will be awarded, but you must show your work.

No calculators, books, notes, or other aids are permitted.

Place a box around your FINAL ANSWER to each question where appropriate.

If you need extra space, you can use the back sides of the pages. Please make it obvious when you have done so.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	10	
2	10	
3	15	
4	15	
5	15	
6	10	
7	10	
Extra Credit	3	
Total	85	

1. (10 points)

a. Find the linearization of $f(x) = \dfrac{1}{x}$ at $a = 10$.

$f'(x) = \dfrac{-1}{x^2}$

$$\boxed{L(x) = f(a) + f'(a)(x-a) = \frac{1}{10} - \frac{1}{100}(x-10)}$$

b. Use your work in part **a.** to approximate $\dfrac{1}{11}$.

$$\frac{1}{11} = f(11) \approx L(11) = \frac{1}{10} - \frac{1}{100}(11-10) = \frac{1}{10} - \frac{1}{100}$$

$$= \frac{9}{100}$$

2. (10 points)

Consider the function

$$f(x) = x + \frac{1}{4x}.$$

a. Find the critical number(s) of $f(x)$.

$$f'(x) = 1 - \frac{1}{4}x^{-2} = 0$$

$$x^2 = \frac{1}{4}$$

$$x = \pm\frac{1}{2}$$

b. Find the absolute maximum and absolute minimum values of $f(x)$ on the interval $[1, 3]$.

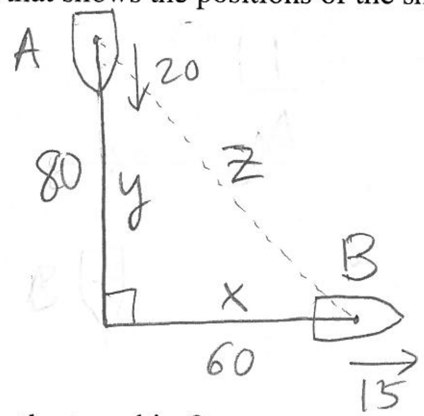
x	$f(x)$	
1	$1+\frac{1}{4} = 5/4$	$\leftarrow$ abs. min
no crit. #s in here		
3	$3+\frac{1}{12} = 37/12$	$\leftarrow$ abs. max.

3. (15 points)

There are two ships at sea, ship A and ship B.

- Ship B is 80 miles south and 60 miles east of ship A.
- Ship A is traveling south at 20 miles per hour.
- Ship B is traveling east at 15 miles per hour.

a. Make a diagram that shows the positions of the ships and their velocities.



b. How far apart are the two ships?

$$z = \sqrt{80^2 + 60^2} = 10\sqrt{64+36} = 100 \text{ miles}$$

(at that instant)

c. Determine the rate at which the distance between the ships changing.

$$x^2 + y^2 = z^2 \leftarrow \text{always true}$$

$$\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} = \cancel{2}z\frac{dz}{dt}$$

$$\frac{dz}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{d} = \frac{60(15) + 80(-20)}{100} = -7\ \frac{\text{miles}}{\text{hr}}$$

(at instant)

4. (15 points)

For the function

$$g(x) = e^{-x^2}$$

a. Find the critical numbers.

$$g'(x) = -2x\,e^{-x^2} = 0$$

$$x = 0 \;\Leftarrow \text{only critical number}$$

b. Find the intervals of increase and decrease.

g': $+$ | $-$ on the x-axis, split at 0

increasing on $(-\infty, 0)$

decreasing on $(0, \infty)$

c. Find all inflection points.

$$g''(x) = -2e^{-x^2} - 2x(-2x)\,e^{-x^2}$$

$$= -2e^{-x^2}(1 - 2x^2) = 0$$

$$2x^2 = 1$$

two inflection points $\rightarrow$ $x = \pm \frac{1}{\sqrt{2}}$

Continued....

Problem 4 continued....

Recall that $g(x) = e^{-x^2}$...

d. Find all local maxima and minima of $g(x)$. (Use a derivative test to justify these.)

only one critical number:

$x = 0$ is local maximum

$g(0) = 1$

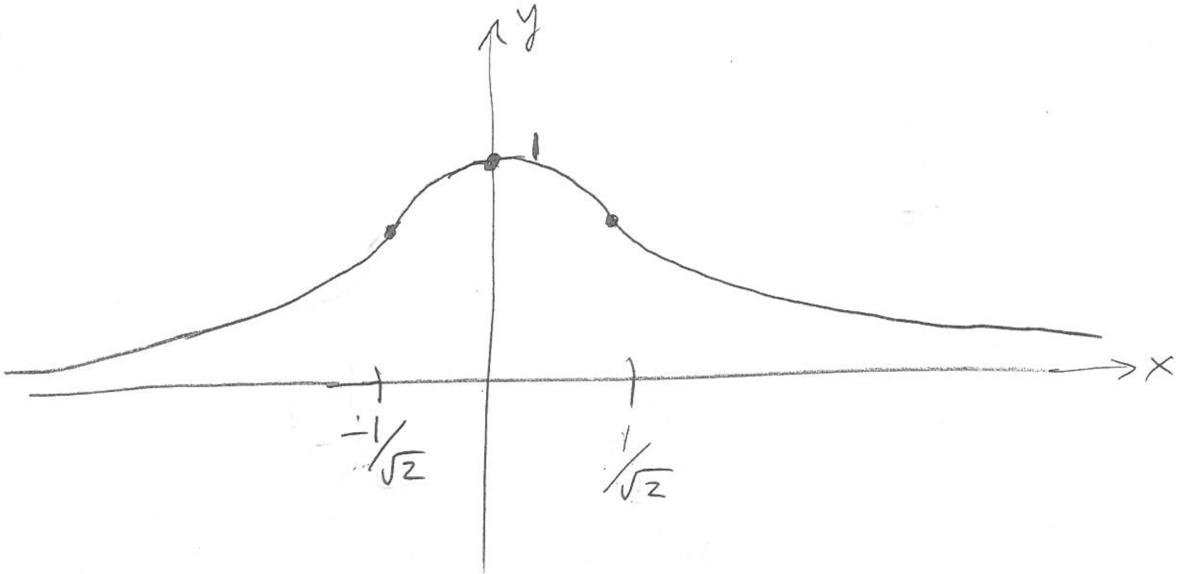
because $g''(0) = -2e^0(1-0) = -2 < 0$

(second-derivative test)

e. State a limit which shows that $y = 0$ is a horizontal asymptote.

$\lim_{x \to +\infty} e^{-x^2} = 0$ (or $\lim_{x \to -\infty} e^{-x^2} = 0$)

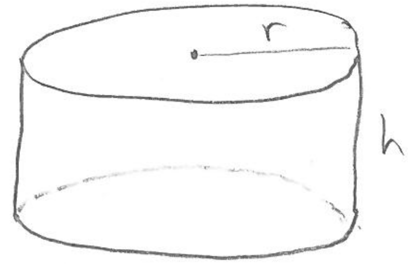
f. Sketch the graph.



5. (15 points)

A cylindrical can with volume 250 cubic centimeters is to be constructed out of aluminum. The walls of the can are made of a thin aluminum that costs 0.2 cents per square centimeter. The lid and bottom are made of a thicker aluminum that costs 0.4 cents per square centimeter. What are the dimensions of the can that minimize its materials costs?

You may find it helpful to recall that for a cylinder with radius r and height h, its volume is $\pi r^2 h$ and its surface area is $2\pi r^2 + 2\pi r h$.

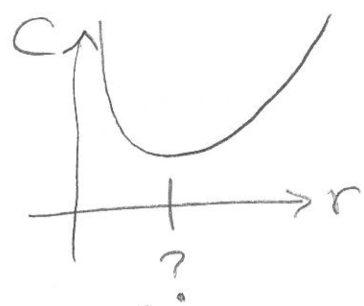


$$C = 0.2(2\pi r h) + 0.4(2\pi r^2)$$

$$250 = \pi r^2 h$$

$$\Longleftrightarrow h = \frac{250}{\pi r^2}$$

$$C(r) = 0.2\left(2\cancel{\pi}\cancel{r}\frac{250}{\cancel{\pi} r^{\cancel{2}}}\right) + 0.4(2\pi r^2)$$



$$= \frac{100}{r} + 0.8\pi r^2$$

$$0 = C'(r) = -100 r^{-2} + 1.6\pi r$$

$$100 = 1.6\pi r^3$$

$$r = \sqrt[3]{\frac{100}{1.6\pi}} \text{ cm}$$

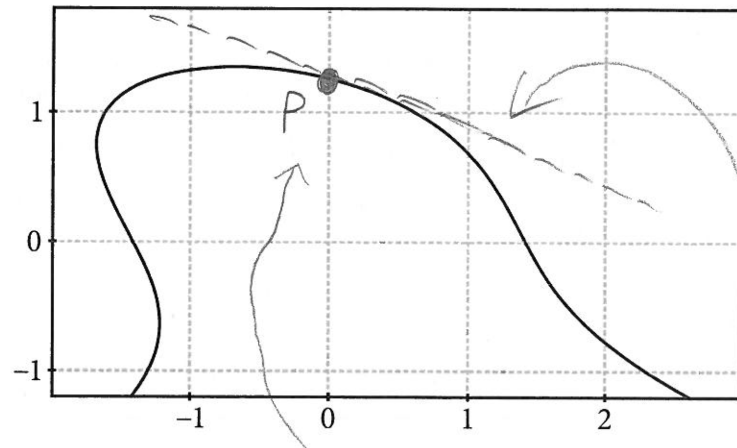
$$h = \frac{250}{\pi r^2} = \frac{250}{\pi\left(\frac{100}{1.6\pi}\right)^{2/3}} \text{ cm}$$

6. (10 points)

Consider the implicitly defined curve given by

$$x^2 + xy + y^3 = 2.$$

A portion of the curve is graphed to the right.



a. Show that the point $P = (0, \sqrt[3]{2})$ is on the curve. Then **draw and label** the point P in the figure.

$x=0$ & $y=\sqrt[3]{2}$: $0^2 + 0\cdot\sqrt[3]{2} + (\sqrt[3]{2})^3 = 2$ ✓

b. Compute y' at P.

$$2x + 1\cdot y + x\cdot\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$\Leftrightarrow \quad y' = \frac{dy}{dx} = \frac{-2x-y}{x+3y^2}$$

$$y'\big|_P = \frac{-2\cdot 0 - \sqrt[3]{2}}{0+3(\sqrt[3]{2})^2} = \frac{-1}{3\sqrt[3]{2}}$$

c. Find the equation of the tangent line at P. Then **draw** this tangent line in the figure.

$$y - \sqrt[3]{2} = \left(\frac{-1}{3\sqrt[3]{2}}\right)(x-0)$$

7. (10 points)

Evaluate the following limits. [Note: You should be careful to apply L'Hôpital's rule **only** when appropriate.]

a. $\lim_{t\to 0} \frac{\sin(t^2)}{t^2}$ $\underset{L'H.}{\overset{\frac{0}{0}}{=}} \lim_{t\to 0} \frac{\cos(t^2)\cdot \cancel{2t}}{\cancel{2t}} = \lim_{t\to 0} \cos(t^2)$

$= 1$

b. $\lim_{x\to 0^+} \sqrt{x}\ln(x)$ $\overset{0\cdot\infty}{=} \lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}} \underset{L'H.}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}}$

$= \lim_{x\to 0^+} -2x^{1/2} = 0$

8. (Extra Credit: 3 points)

Use the fact that $\frac{d}{dx}b^x = \ln(b)\, b^x$ to show that

$$\frac{d}{dx}\log_b(x) = \frac{1}{x\ln(b)}.$$

(Hint: Rewrite $y = \log_b(x)$ in a way that allows you to use the given fact.)

$y = \log_b(x)$

$b^y = x$

$\frac{d}{dx}$: $\ln(b)\, b^y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\ln(b)\, b^y}$

$= \frac{1}{\ln(b)\, x}$