Your Name

Solutions

Your Signature

Instructor Name

Problem	Total Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Extra Credit	(5)	
Total	100	

- This test is closed notes and closed book.
- You may **not** use a calculator.
- In order to receive full credit, you must **show your work.** Be wary of doing computations in your head. Instead, write out your computations on the exam paper.
- Raise your hand if you have a question.

1. (10 points) Let $f(x) = x^2 - \frac{5}{x}$.

(a) Using the **definition of the derivative**, find $f'(x)$. No credit will be given if a different method is used. [It is recommended you start by writing the definition of the derivative.]

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - \frac{5}{x+h} - \left(x^2 - \frac{5}{x}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2} + \frac{5}{x} - \frac{5}{x+h}}{h}$$

$$= \lim_{h\to 0} \left[\frac{2xh+h^2}{h} + \frac{1}{h}\left(\frac{5x+5h-5x}{x(x+h)}\right)\right]$$

$$= \lim_{h\to 0} \left[2x + h + \frac{5}{x(x+h)}\right]$$

$$= \boxed{2x + \frac{5}{x^2}}$$

(b) Write an equation of the line tangent to the graph of $f(x)$ when $x = 2$.

$f(2) = 2^2 - \frac{5}{2} = 4 - \frac{5}{2} = \frac{8-5}{2} = \frac{3}{2}$; point $\left(2, \frac{3}{2}\right)$

$f'(2) = 2\cdot 2 + \frac{5}{2^2} = 4 + \frac{5}{4} = \frac{16+5}{4} = \frac{21}{4} = m$

line: $y - \frac{3}{2} = \frac{21}{4}(x-2)$. So $\boxed{y = \frac{3}{2} + \frac{21}{4}(x-2)}$

2. (10 points) During a storm, snow is falling on a mountain at a rate of

$$M(t) = t^2 - \frac{t^3}{3}$$

feet per hour for a three hour period starting at time $t = 0$.

(a) Determine the *net change* in the height of snow during the first two hours of the storm. Include units with your answer.

$$\int_0^2 \left(t^2 - \frac{1}{3}t^3\right) dt = \frac{1}{3}t^3 - \frac{1}{12}t^4 \Big]_0^2 = \frac{8}{3} - \frac{16}{12} = \frac{8}{3} - \frac{4}{3}$$

$$= \boxed{\frac{4}{3} \text{ feet}}$$

(b) Determine the height of the snow on the mountain or explain why this is not possible with the present information.

It's not possible to determine the height because we don't know how much snow was present when the storm started at $t=0$

(c) Observe that $M(2.5) > 0$ and $M'(2.5) < 0$. Explain what these two facts indicate about the snow falling when $t = 2.5$.

At $t=2.5$, the height of the snow is increasing (since $M(2.5)>0$) but it's increasing at a slower rate (since $M'(2.5)<0$).
In short, the storm, while still going, is dissipating.

3. (10 points) The population of ants in a new colony is modeled by the function

$$p(t) = 500\left(t + \frac{1}{2}\ln(1+t^2)\right) + 200,$$

where t is measured in months.

(a) Find $p(0)$ and interpret in the context of the problem.

$p(0) = 500(0 + \frac{1}{2}\ln(1)) + 200 = 0 + 200$

The colony starts with 200 ants.

(b) Find $\lim_{t\to\infty} p(t)$ and interpret in the context of the problem.

$\lim_{t\to\infty} 500\left(t + \frac{1}{2}\ln(1+t^2)\right) + 200 = \infty$

The ants take over the world by this model.

(c) Find $p'(t)$.

rewrite $p(t) = 500t + 250\ln(1+t^2) + 200$

$p'(t) = 500 + 250\left(\frac{1}{1+t^2}\right)(2t) = 500 + \frac{500t}{1+t^2}$

(d) Find $p'(1)$ and interpret in the context of the problem.

$p'(1) = 500 + \frac{500}{2} = 750$ ants/month

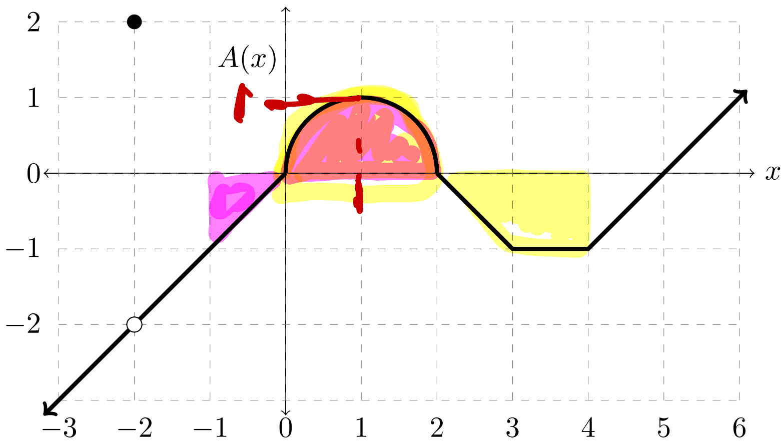
The colony is increasing by 750 ants/month after 1 month.

(e) Find $\lim_{t\to\infty} p'(t)$ and interpret in the context of the problem.

$\lim_{t\to\infty}\left(500 + \frac{500t}{1+t^2}\right) = 500$ ants/month

In the long term, the population of the colony increases at the rate of 500 ants per month.

4. (10 points) Consider the function $A(x)$ graphed below. Between $x = 3$ and $x = 5$, the graph is of a semicircle of radius 1.



(a) $\lim_{x \to -2} A(x) =$ -2

(b) $A(-2) =$ 2

(c) $A'(-1) =$ 1

(d) At what x values, if any, does $A'(x)$ not exist?

$x = -2, 0, 2, 3, 4$

(e) Evaluate $\int_{-1}^{2} A(x)\,dx$. $= -\frac{1}{2} + \frac{\pi}{2} = \frac{\pi - 1}{2}$

(f) Let $H(x) = \int_0^x A(s)\,ds$. What is the value of $H(4)$?

$= \frac{\pi}{2} - \frac{3}{2} = \frac{\pi - 3}{2}$

(g) For $H(x)$ from part **f.**, what is the value of $H'(1)$.

1

5. (10 points) Sketch the graph of a function that satisfies **all** of the given conditions. **Label all important items in your graph.**

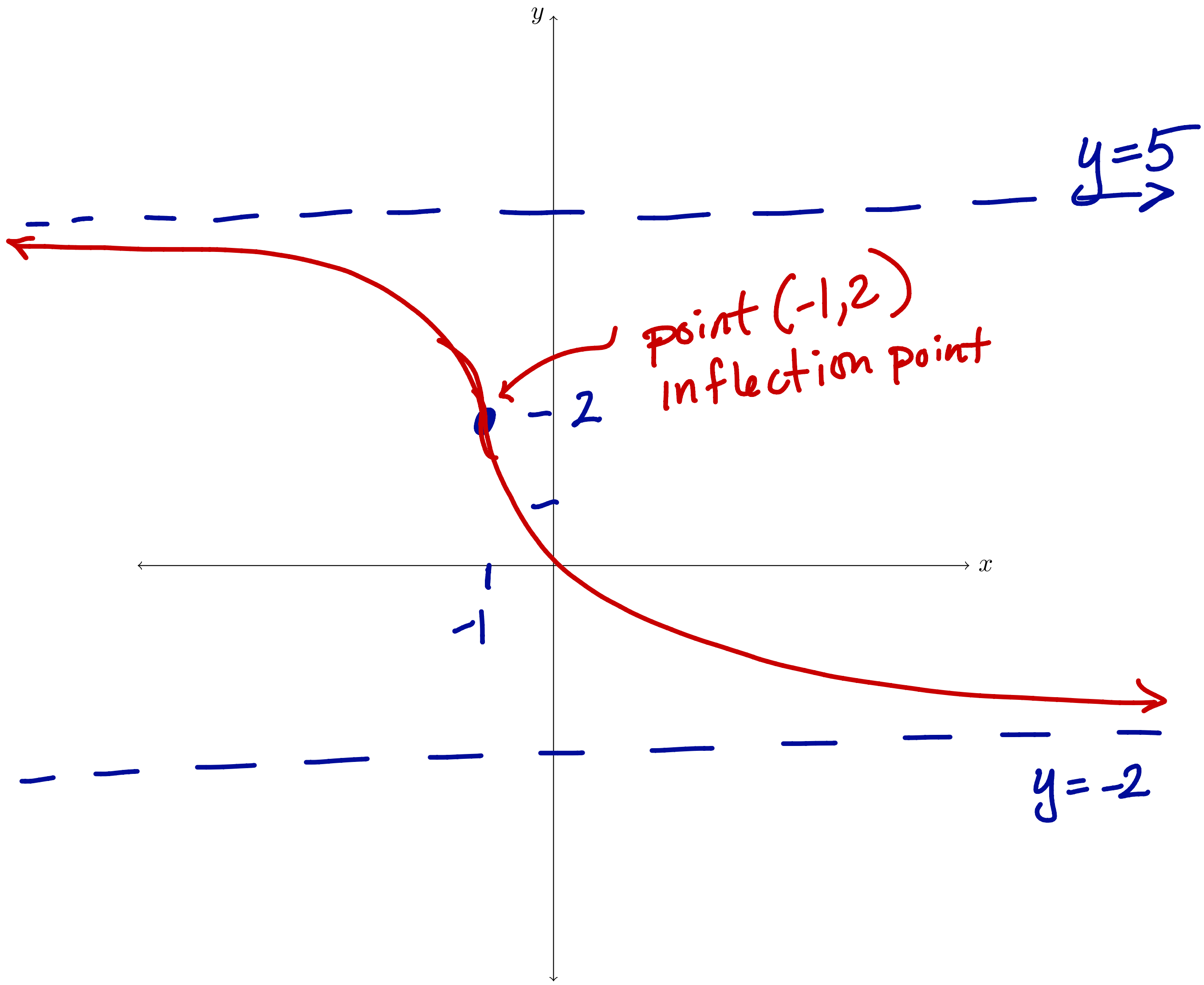
(a) The domain of f is $(-\infty, \infty)$.

(b) $f'(x) < 0$ if $x \neq -1$. ← f is always decreasing

(c) $f''(x) < 0$ if $x < -1$ and $f''(x) > 0$ if $x > -1$. ccdown to ccup. Switch at $x=-1$

(d) $f(-1) = 2$ point $(-1,2)$

(e) $\lim_{x\to-\infty} f(x) = 5$ and $\lim_{x\to\infty} f(x) = -2$ asymptotes (horizontal) $y=5$, $y=-2$



6. (10 points) Differentiate the following functions. For parts (a) and (b), it is not necessary to simplify your answers.

(a) $f(x) = \dfrac{e^{2x}}{\sqrt{x^2+1}} + \arctan(3x)$

$$f'(x) = \frac{(x^2+1)^{1/2}(2e^{2x}) - e^{2x}\left(\frac{1}{2}(x^2+1)^{-1/2}(2x)\right)}{x^2+1} + \frac{3}{1+9x^2}$$

$$= \frac{e^{2x}\left(2\sqrt{x^2+1} - \frac{x}{\sqrt{x^2+1}}\right)}{x^2+1} + \frac{3}{1+9x^2}$$

(b) $g(x) = \displaystyle\int_x^3 2t\sin(t^2)\,dt$ $= -\displaystyle\int_3^x 2t\sin(t^2)\,dt$

$$g'(x) = -\left(2x\sin(x^2)\right)$$

(c) Find $\frac{dy}{dx}$ by implicit differentiation: $\ln(xy) - \cos y = ye^x$

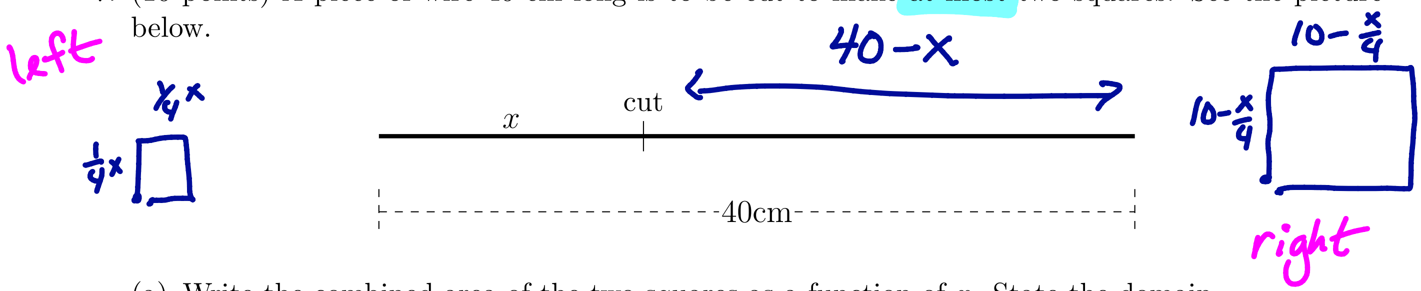
$$\frac{1}{xy}\left(y + x\frac{dy}{dx}\right) + (\sin y)\frac{dy}{dx} = ye^x + \frac{dy}{dx}e^x$$

$$\frac{1}{x} + \left(\frac{1}{y}\right)\frac{dy}{dx} + (\sin y)\frac{dy}{dx} = ye^x + e^x\frac{dy}{dx}$$

$$\left(\frac{1}{y} + \sin y - e^x\right)\frac{dy}{dx} = ye^x - \frac{1}{x}$$

$$\boxed{\frac{dy}{dx} = \left(ye^x - \frac{1}{x}\right)\Big/\left(\frac{1}{y} + \sin y - e^x\right)}$$

7. (10 points) A piece of wire 40 cm long is to be cut to make at most two squares. See the picture below.



(a) Write the combined area of the two squares as a function of x. State the domain.

$$A(x) = \frac{1}{16}x^2 + \left(10 - \frac{x}{4}\right)^2 \quad ; \qquad \text{domain: } [0, 40]$$

left square

right square

(b) For what value(s) of x does the area function have a potential maximum or minimum?

$$A'(x) = \frac{1}{8}x + 2\left(10 - \frac{x}{4}\right)\left(-\frac{1}{4}\right) = \frac{1}{8}x - \frac{1}{2}\left(10 - \frac{x}{4}\right)$$

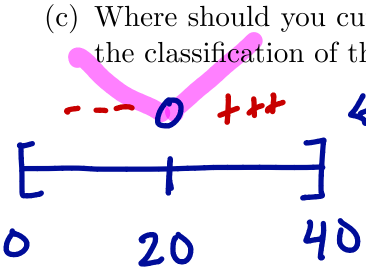
$$= \frac{1}{8}x - 5 + \frac{x}{8} = \frac{x}{4} - 5 = 0$$

$\frac{x}{4} = 5$ or $x = 20$

$$\boxed{\text{ANSWER to question: } x = 0, 20, 40}$$

(c) Where should you cut to minimize the combined area? Maximize the combined area? Justify the classification of the extrema.

look here



← sign A'

First derivative Test indicates a local min at $x=20$. It's unique; so it's absolute.

Ends must be maximum. By symmetry, both give equal squares

ANSWER:

For minimum, cut at $x=20$

For maximum, don't cut. (or $x=0$ and $x=40$.)

8. (10 points)

A particle is moving with acceleration

$$a(t) = t + e^{t/2}.$$

You measure that at time $t = 0$, its position is given by $s(0) = 4$ and its velocity is given by $v(0) = 6$

Determine the position function $s(t)$.

$$v(t) = \int a(t)\,dt = \int \left(t + e^{t/2}\right) dt = \frac{1}{2}t^2 + 2e^{t/2} + C$$

$$6 = v(0) = 0 + 2 + C. \text{ So } C = 4$$

$$\boxed{v(t) = \frac{1}{2}t^2 + 2e^{t/2} + 4}$$

$$s(t) = \int v(t)\,dt = \int \left(\frac{1}{2}t^2 + 2e^{t/2} + 4\right) dt$$

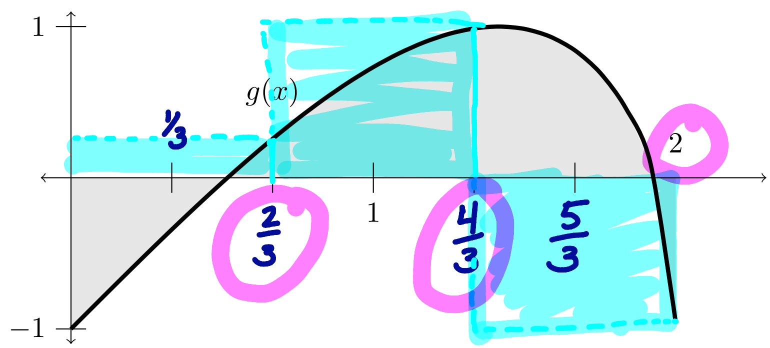
$$= \frac{1}{6}t^3 + 4e^{t/2} + 4t + C$$

$$4 = s(0) = 0 + 4 + 0 + C. \quad \text{So } C = 0.$$

$$\boxed{s(t) = \frac{1}{6}t^3 + 4e^{t/2} + 6t}$$

9. (10 points)

Consider the function $g(x) = x\sqrt{4-x^2} - 1$, part of whose graph is shown below.



(a) Write down, but do not evaluate, a computation that approximates $\int_0^2 g(x)\,dx$ using three **right-hand** rectangles. $\boxed{\text{Draw and shade}}$ the rectangles on the graph. (No need to simplify your answer.)

$$\int_0^2 g(x)\,dx \approx \frac{2}{3}\left(\left(\frac{2}{3}\sqrt{4-\left(\frac{2}{3}\right)^2}-1\right)+\left(\frac{4}{3}\sqrt{4-\left(\frac{4}{3}\right)^2}-1\right)+(-1)\right)$$

(b) Determine $\int_0^2 g(x)\,dx$ exactly. Show your work, and simplify your answer.

$$\int_0^2 \left(x\sqrt{4-x^2}-1\right)dx = \int_0^2 (4-x^2)^{1/2}x\,dx - \int_0^2 1\,dx$$

$$= -\frac{1}{3}(4-x^2)^{3/2} - x\Big]_0^2 = (0-2)-\left(-\frac{1}{3}(4)^{3/2}-0\right)$$

$$= -2+\frac{8}{3} = \frac{2}{3}$$

(c) How could you make the estimate in (a) more accurate?

more rectangles.

10. (10 points) Newton's method can be used to find an approximate solution to the equation $x^2 = 8$. To apply Newton's method to find these roots, let $f(x) = x^2 - 8$.

(a) Use Newton's method with initial approximation $x_0 = 2$ to find x_1, a better estimate of a root of the given equation.

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \; ; \quad f'(x) = 2x$$

$$x_1 = x_0 - \frac{x_0^2 - 8}{2x_0} = 2 - \frac{2^2 - 8}{2 \cdot 2} = 2 - \frac{-4}{4} = 2 + 1 = 3$$

(b) Apply one more iteration of Newton's method to find x_2.

$$x_2 = x_1 - \frac{x_1^2 - 8}{2x_1} = 3 - \frac{3^2 - 8}{2 \cdot 3} = 3 - \left(\frac{1}{6}\right)$$

$$= \frac{17}{6}$$

(c) Notice the equation $x^2 = 8$ has two roots. What value of x_0 would make a good choice to find the **other** root?

Pick $x_0 = -2$.

Extra Credit (up to 5 points):

(a) Show that the equation $\cos x - 2x = 4$ has at least one real solution.

Consider $f(x) = \cos x - 2x - 4$

When $x = 0$, $f(0) = 1 - 0 - 4 = -3 < 0$

When $x = -4\pi$, $f(-4\pi) = 1 - 2(-4\pi) - 4 = 8\pi - 3 > 0$

Since $f(x)$ is continuous and positive at $x = -4\pi$ negative at $x = 0$, it must be equal to zero somewhere on the interval $(-4\pi, 0)$ by the Intermediate Value Theorem.

(b) Show that the equation $\cos x - 2x = 4$ has exactly one real solution. (Hint: what does the derivative tell you about the behavior of the function $g(x) = \cos(x) - 2x - 4$?)

$g'(x) = -\sin x - 2$.

Since $-\sin x \leq 1$, we know $g'(x) = -\sin x - 2 \leq -1$.

So $g'(x)$ is always negative.

So $g(x)$ is always decreasing.

So $g(x)$ can have at most one root.