Math F251 — Midterm 2 — Fall 2021

Name: Solutions

Section:
- ☐ F01 (Jill Faudree)
- ☐ F02 (James Gossell)
- ☐ UX1 (James Gossell)

Rules:

You have 60 minutes to complete the exam.

Partial credit will be awarded, but you must show your work.

The exam is closed book and closed notes.

Calculators are not allowed.

Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question where appropriate.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	12	
2	6	
3	12	
4	20	
5	12	
6	8	
7	10	
8	10	
9	10	
Total	100	

1. (12 points) The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$ where r is the radius and h is the height of the cone.

   (a) Assuming that the radius and the height are changing with time, t, find $\frac{dV}{dt}$.

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = \frac{\pi}{3}\left[2r\frac{dr}{dt}h + r^2\frac{dh}{dt}\right]$$

   (b) If the height of the cone decreases by 1 inch per hour while the radius increases by 1 inch per hour, find the rate of change of the volume of the cone at the moment when the radius is 10 inches and the height is 5 inches. Include units with your answer. Interpret your answer in the context of the problem

$\frac{dh}{dt} = -1$ in/hr, $\frac{dr}{dt} = 1$ in/hr. Find $\frac{dV}{dt}$ when $r = 10$ in and $h = 5$ in

Plug into

$$\frac{dV}{dt} = \frac{\pi}{3}\left[2 \cdot 10 \cdot 1 \cdot 5 + 10^2(-1)\right] = \frac{\pi}{3}\left[100 - 100\right] = 0 \text{ in}^3/\text{hr}.$$

Interpetation: The volume is constant at this moment.

2. (6 points) The radius of a spherical ball is measured to be 5 inches with a possible error of ±0.5 inch. Use differentials to estimate the maximum possible error in the volume of the ball.
(Note: The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$ where r is the radius of the sphere.)

$$V = \frac{4}{3}\pi r^3$$

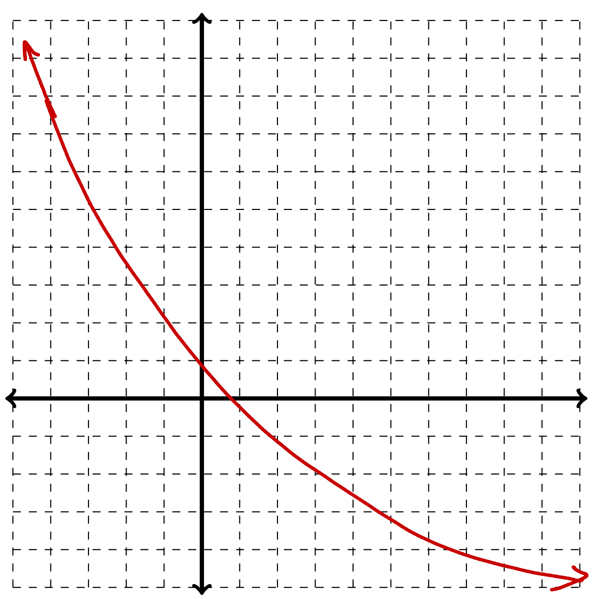
$$dv = \frac{4}{3}\pi \cdot 3r^2\, dr = 4\pi r^2 dr$$

$r = 5$ in
$dr = 0.5$ in

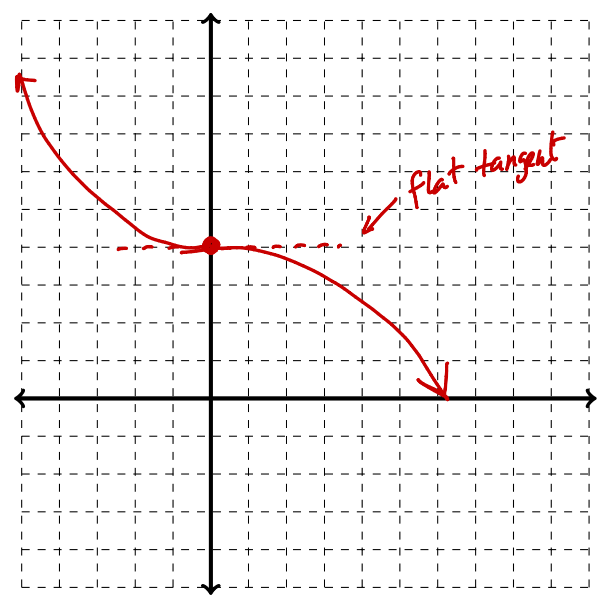
$$dv = 4\pi \cdot 5^2 \cdot \left(\tfrac{1}{2}\right) = 50\pi \text{ in}^3$$

3. (12 points)

(a) Sketch a graph of a function $f(x)$ such that
$f'(x) < 0$ and decreasing
$f''(x) > 0$ c cup
for all real numbers x.



(b) Sketch a graph of a function $f(x)$ such that
$f'(0) = 0$, flat tangent
$f''(x) > 0$ when $x < 0$, and c cup
$f''(x) < 0$ when $x > 0$. c c down



4. (20 points) Use the information below to answer questions about the function $f(x)$. Make sure you answer the question!

$$f(x) = \frac{2x^2 - 1}{x^2 + 3}, \qquad f'(x) = \frac{14x}{(x^2 + 3)^2}, \qquad f''(x) = \frac{-42(x^2 - 1)}{(x^2 + 3)^3}.$$

(a) Determine the intervals on which $f(x)$ is increasing/decreasing.

$f' = 0$ when $x = 0$

$- -$ 0 $+ +$ ← sign f'

0

f' is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$

(b) Find the local maximum/minimum values of $f(x)$. If something doesn't exist, you must explicitly state this and justify your answer.

f has a local min at $x=0$. The min. value is $f(0) = -\frac{1}{3}$

f has no local max as f has only one crit. pt.

(c) Find the intervals on which $f(x)$ is concave up and concave down.

$f'' = 0$ when $x = \pm 1$

$- - -$ 0 $+ + +$ 0 $- - -$ ← sign f''

-1 $\quad$ 1

f is ccup on $(-1, 1)$

f is ccdown on $(1, \infty) \cup (-\infty, -1)$

(d) Find any inflection points of $f(x)$. If there aren't any, you must explicitly state this and justify your answer.

inflection pts: $x = 1$, $f(x) = \frac{1}{4}$

$x = -1$, $f(x) = \frac{1}{4}$

answer $(\pm 1, \frac{1}{4})$

(e) Find any horizontal asymptotes of $f(x)$ or state that none exist. Justify your answer.

claim: $y = 2$ is a horizontal asymptote.

justification: $\lim\limits_{x \to \infty} \frac{2x^2 - 1}{x^2 + 3} \cdot \frac{1/x^2}{1/x^2} = \lim\limits_{x \to \infty} \frac{2 - \frac{1}{x^2}}{1 + 3/x^2} = \frac{2-0}{1+0} = 2.$

5. (12 points) Find the pair of numbers x and y on the line $3x+y=15$ such that the product $(x+2)(y-3)$ is as large as possible.

   (a) Explicitly state the quantity you want to maximize or minimize.

   (b) Identify the domain of your function.

   (c) Identify your answer. (Note: Your answer may not be an integer.)

   (d) Justify that your answer is correct. That is, use Calculus to show that your answer indeed does represent a maximum or minimum.

(a) maximize $P=(x+2)(y-3)$

$y=15-3x$

$P(x)=(x+2)(15-3x-3)=(x+2)(12-3x)=-3(x^2-2x-8)$

(b) domain: $(-\infty,\infty)$

$P'(x)=-3(2x-2)=0 \quad x=1$ (crit. pt.)

(d) Justification: (many options) • P is a downward openning parabola.

So its one cr. pt. is an abs. max.

OR

• Sign of P': $+++\;0\;---$ at 1

So P has an abs max b/c it's a local max and unique

• $P''=-6<0$. So P is cc down. So P has abs max at $x=1$ b/c x is a local max and unique.

(c) Answer: The closest pair is $x=1$, $y=12$

6. (8 points) Evaluate the following limit using L'Hopital's Rule. Before each application of L'Hopital's Rule, you must indicate the form of the limit (0/0 or ∞/∞).

form $\frac{0}{0}$

$$\lim_{x\to 0^+} \frac{\ln(\sin x + \cos x)}{x}$$

$$\lim_{x\to 0^+} \frac{\ln(\sin x+\cos x)}{x} \overset{H}{=} \lim_{x\to 0^+} \frac{\frac{\cos x - \sin x}{\sin x+\cos x}}{1} = \lim_{x\to 0^+} \frac{\cos x - \sin x}{\sin x + \cos x}$$

$$= \frac{1-0}{0+1} = 1$$

7. (10 points) Evaluate the following indefinite integrals:

(a) $\int x(x-1)\,dx = \int (x^2-x)\,dx = \frac{1}{3}x^3 - \frac{1}{2}x^2 + C$

(b) $\int \left(\sec^2 x - e^x + \frac{1}{\sqrt{1-x^2}}\right) dx = \tan(x) - e^x + \arcsin(x) + C$

8. (10 points) A drone is released into the air from an initial height of 5 feet off the ground. Its upward velocity at t seconds is $v(t) = 3t^2 - 3$ feet per second. How high will the drone be after 2 seconds?

$s(0)=5$

$v(t)=3t^2-3$

$s(t)=\int v(t)\,dt = \int (3t^2-3)\,dt = t^3 - 3t + C$

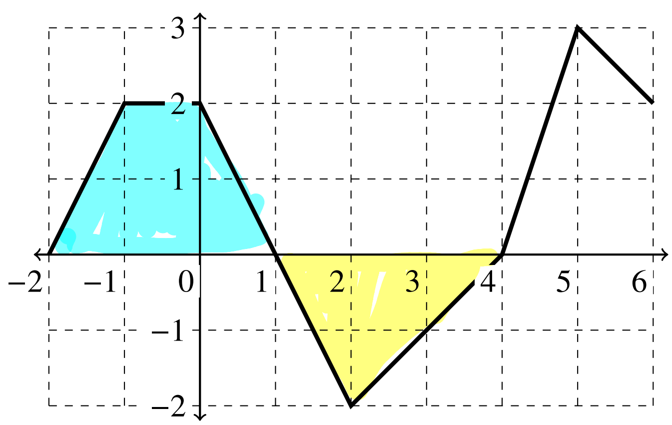
$5 = s(0) = 0^3 - 3\cdot 0 + C$

So $C=5$

$s(t) = t^3 - 3t + 5$

$s(2) = 2^3 - 6\cdot 2 + 5 = 8 - 12 + 5 = 1$ ft

9. (10 points) Using the graph of $f(x)$ shown (below) and geometry, calculate exactly each of the following quantities. Show your work to receive partial credit.



(a) $\int_{-2}^{4} f(x)\,dx = 4 - 3 = 1$

(b) $\int_{-2}^{1} (2f(x)+5)\,dx = 2\int_{-2}^{1} f(x)\,dx + \int_{-2}^{1} 5\,dx$

$= 2(4) + 5(3) = 8 + 15 = 23$