Math F251 Midterm 2 Fall 2021

Name: Solutions

Section:
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Rules:

You have 60 minutes to complete the exam.

Partial credit will be awarded, but you must show your work.

The exam is closed book and closed notes.

Calculators are not allowed.

Place a box around your FINAL ANSWER to each question where appropriate.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	6	
2	12	
3	12	
4	20	
5	12	
6	8	
7	10	
8	10	
9	10	
Total	100	

1. (6 points) The radius of a spherical ball is measured to be 3 inches with a possible error of ±0.25 inch. Use differentials to estimate the maximum possible error in the volume of the ball.
(Note: The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$ where r is the radius of the sphere.)

$$V = \frac{4}{3}\pi r^3$$

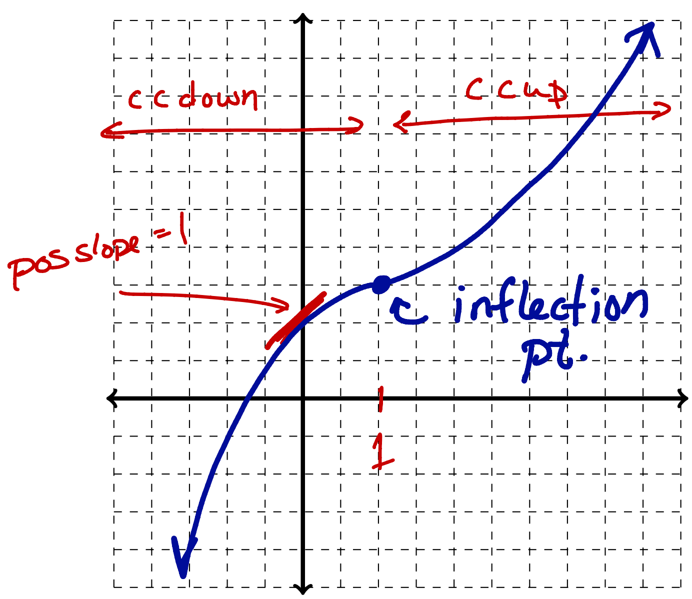
$$dV = \frac{4}{3} \cdot \pi \cdot 3r^2 \, dr = 4\pi r^2 \, dr$$

$$r = 3, \quad dr = \frac{1}{4}$$

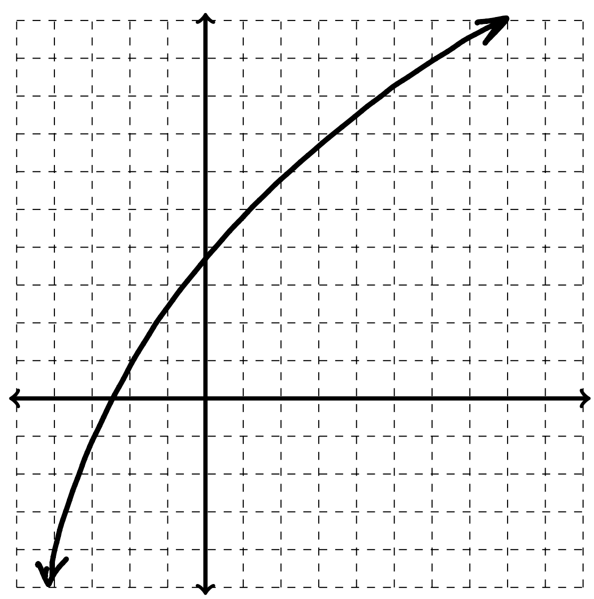
$$dV = 4\pi \cdot 3^2 \cdot \frac{1}{4} = 9\pi \text{ in}^3$$

2. (12 points)

(a) Sketch a graph of a function $f(x)$ such that
$f'(0) = 1$, pos. slope at $x=0$
$f''(x) < 0$ when $x < 1$, and cc down
$f''(x) > 0$ when $x > 1$. cc up



(b) Sketch a graph of a function $f(x)$ such that
$f'(x) > 0$ and increasing
$f''(x) < 0$ cc down
for all real numbers x.



3. (12 points) The volume of a cylinder is given by $V = \pi r^2 h$ where r is the radius and h is the height of the cylinder.

   (a) Assuming that the radius and the height are changing with time, t, find $\frac{dV}{dt}$.

$$V = \pi r^2 h$$

$$\frac{dV}{dt} = \pi\left(2r\frac{dr}{dt}h + r^2\frac{dh}{dt}\right)$$

   (b) If the height of the cylinder decreases by 4 inches per hour while the radius increases by 1 inch per hour, find the rate of change of the volume of the cylinder at the moment when the radius is 5 inches and the height is 10 inches. Include units with your answer. Interpret your answer in the context of the problem

$$\frac{dh}{dt} = -4 \qquad \frac{dr}{dt} = 1$$

Find $\frac{dV}{dt}$ when $r=5$ and $h=10$

$$\frac{dV}{dt} = \pi\left(2\cdot 5\cdot 1\cdot 10 + 5^2\cdot(-4)\right)$$

$$= \pi(100 - 100) = 0 \ \text{in}^3/\text{hr}$$

Interpretation: The volume is constant at this moment.

4. (20 points) Use the information below to answer questions about the function $f(x)$. Make sure you answer the question!

$$f(x) = \frac{1-3x^2}{x^2+3}, \qquad f'(x) = \frac{-20x}{(x^2+3)^2}, \qquad f''(x) = \frac{60(x^2-1)}{(x^2+3)^3}.$$

(a) Determine the intervals on which $f(x)$ is increasing/decreasing.

$f' = 0$ when $x = 0$

$+ + + \ 0 \ - - -$ ← f'

0

$f(x)$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

(b) Find the local maximum/minimum values of $f(x)$. If something doesn't exist, you must explicitly state this and justify your answer.

f has a local max at $x=0$; its maximum value is $f(0) = \frac{1}{3}$.

f has no local min because there are no other crit. pts.

(c) Find the intervals on which $f(x)$ is concave up and concave down.

$f'' = 0$ when $x = \pm 1$.

$+ + + \ 0 \ - - - \ 0 \ + + +$ ← sign of f''

$-1 \qquad 1$

f is ccup on $(-\infty, -1) \cup (1, \infty)$

ccdown on $(-1, 1)$

(d) Find any inflection points of $f(x)$. If there aren't any, you must explicitly state this and justify your answer.

f has inflection points when $x=-1$, $f(-1) = \frac{-1}{2}$; $x = 1$, $f(1) = \frac{-1}{2}$

points $\left(\pm 1, \frac{-1}{2}\right)$

(e) Find any horizontal asymptotes of $f(x)$ or state that none exist. Justify your answer.

claim: $y = -3$ is a horizontal asymptote

justification: $\displaystyle \lim_{x\to\infty} \frac{1-3x^2}{x^2+3} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{\frac{1}{x^2} - 3}{1 + \frac{3}{x^2}} = -3$

5. (12 points) Find the pair of numbers x and y on the line $3x+y = 15$ such that the product $(x+2)(y+3)$ is as large as possible.

   (a) Explicitly state the quantity you want to maximize or minimize.

   (b) Identify the domain of your function.

   (c) Identify your answer. (Note: Your answer may not be an integer.)

   (d) Justify that your answer is correct. That is, use Calculus to show that your answer indeed does represent a maximum or minimum.

ⓐ maximize $P=(x+2)(y+3)$

Substitute: $y=15-3x$ into P: $P(x)=(x+2)(15-3x+3)$

$$=(x+2)(18-3x)$$
$$=-3(x-6)(x+2)$$
$$=-3(x^2-4x-12)$$

ⓑ domain: $(-\infty,\infty)$

$P'(x)=-3(2x-4)=0$ $\quad x=2$ crit. pt

ⓒ Answer: $x=2$, $y=9$

ⓓ <u>Justification</u>: (There are many.)

- P is a parabola that opens down. So x is an abs. max.
- $+ + + 0 - - -$ ← P' (at $x = 2$) $\quad$ P has a local max at $x=2$ + it's unique.
- $P''=-6<0$. So x has a local max at $x=2$ + unique.

6. (8 points) Evaluate the following limit using L'Hopital's Rule. Before each application of L'Hopital's Rule, you must indicate the form of the limit (0/0 or ∞/∞).

$$\lim_{x \to 0^+} \frac{\ln(\sin x + \cos x)}{x}$$

form $\frac{0}{0}$

$$\lim_{x \to 0^+} \frac{\ln(\sin(x) + \cos(x))}{x} \overset{H}{=} \lim_{x \to 0^+} \frac{\frac{\cos(x) - \sin(x)}{\sin(x) + \cos(x)}}{1} = \frac{\frac{1-0}{0+1}}{1} = 1$$

7. (10 points) Evaluate the following indefinite integrals:

(a) $\int x(x+1)\,dx = \int (x^2 + x)\,dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$

(b) $\int \left(\sec(x)\tan(x) - e^x + \frac{1}{1+x^2}\right) dx = \sec(x) - e^x + \arctan(x) + C$

8. (10 points) A drone is released into the air from an initial height of 10 feet off the ground. Its upward velocity at t seconds is $v(t) = 3t^2 - 3$ feet per second. How high will the drone be after 2 seconds?

$S(0) = 10$

$s(t) = \int v(t)\,dt = \int (3t^2 - 3)\,dt = t^3 - 3t + C$

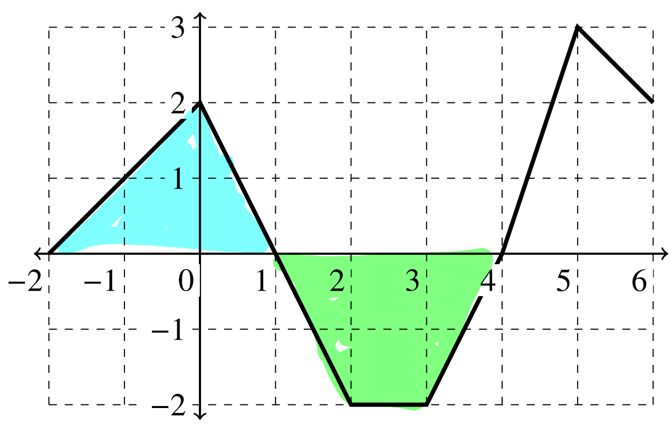
$10 = S(0) = 0^3 - 3 \cdot 0 + C$

$10 = C$

$S(t) = t^3 - 3t + 10$

$S(2) = 2^3 - 3 \cdot 2 + 10 = 8 - 6 + 10 = \underline{12 \text{ feet}}$

9. (10 points) Using the graph of $f(x)$ shown (below) and geometry, calculate exactly each of the following quantities. Show your work to receive partial credit.



(a) $\int_{-2}^{4} f(x)\,dx = 3 - 4 = -1$

(b) $\int_{-2}^{1} (2f(x) + 5)\,dx = 2\int_{-2}^{1} f(x)\,dx + \int_{-2}^{1} 5\,dx = 2 \cdot 3 + 5 \cdot 3 = 6 + 15$

$= 21$