Math F251 Final Exam Fall 2022

Name: Solutions

Rules:

You have 2 hours to complete the final exam.

Partial credit will be awarded, but you must show your work.

You may have a single handwritten 3×5 notecard.

Calculators are not allowed.

Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question where appropriate.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	8	
2	8	
3	8	
4	10	
5	10	
6	10	
7	16	
8	12	
9	12	
10	6	
Extra Credit	5	
Total	100	

1. (8 points) Evaluate the limits. An answer without clear, mathematically precise work, will not earn full credit. Any use of L'Hôpital's Rule should be indicated using an **H** over the equal sign.

algebra (a) $\lim_{x \to -4} \frac{\frac{1}{4} + \frac{1}{x}}{x+4} = \lim_{x \to -4} \left(\frac{1}{x+4}\right)\left(\frac{x+4}{4x}\right) = \lim_{x \to -4} \frac{1}{4x} = \frac{-1}{16}$

L'Hop. $\lim_{x \to -4} \frac{\frac{1}{4} + x^{-1}}{x+4} \overset{H}{=} \lim_{x \to -4} \frac{-x^{-2}}{1} = -(-4)^{-2} = \frac{-1}{16}$

form $\frac{0}{0}$

(b) $\lim_{x \to -1^+} \frac{8 + x^2}{x^2 - 1} = -\infty$

as $x \to -1^+$, $8+x^2 \to 9$ which is positive.
$x^2 - 1 \to 0^-$ which is negative.

So the quotient is negative.

2. (8 points) Find the derivative. You do not need to simplify your answer.

(a) $B(x) = \frac{\tan(2x)}{x^2 + 16}$

$$B'(x) = \frac{(x^2+16)(\sec^2(2x))(2) - (\tan(2x))(2x)}{(x^2+16)^2}$$

(b) $f(x) = e^{-x} + \frac{1}{x} + \sqrt{2x} = e^{-x} + x^{-1} + \sqrt{2} \cdot x^{1/2}$

$$f'(x) = -e^{-x} - x^{-2} + \sqrt{2} \cdot \left(\frac{1}{2}\right) x^{-1/2}$$

3. (8 points) Evaluate the integrals. You do not need to simplify your answer.

(a) $\int (4\sin(x) + 8x^3 - 1)\,dx$ $= -4\cos(x) + \frac{8}{4}x^3 - x + C$

(b) $\int_0^2 \pi x\sqrt{x^2+9}\,dx = \frac{\pi}{2}\int_9^{13} u^{1/2}\,du = \frac{\pi}{2}\cdot\frac{2}{3}\cdot u^{3/2}\Big|_9^{13}$

let $u = x^2 + 9$

$du = 2x\,dx$

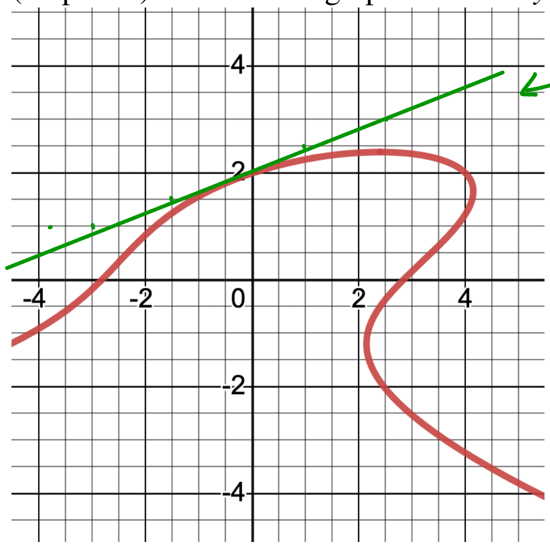
$\frac{1}{2}du = x\,dx$

$= \frac{\pi}{3}\left(13^{3/2} - 9^{3/2}\right)$

if $x=0$, $u=9$

$x=2$, $u=13$

4. (10 points) Below is the graph of $x^2 - 2xy + y^3 = 8$.



tangent line
$m = \frac{1}{3}$

(a) Use implicit differentiation to find $\frac{dy}{dx}$.

$$2x - 2 \cdot y - 2x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$(-2x + 3y^2)\left(\frac{dy}{dx}\right) = 2y - 2x$$

$$\boxed{\frac{dy}{dx} = \frac{2y-2x}{-2x+3y^2}}$$

(b) Find an equation of the line tangent to the curve $x^2 - 2xy + y^3 = 8$ when $x = 0$.

point: $(0,2)$

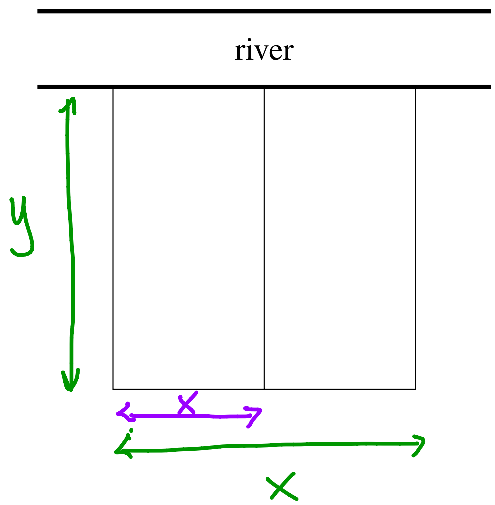
slope: $\frac{dy}{dx} = \frac{2\cdot 2 - 2\cdot 0}{-2\cdot 0 + 3\cdot 2^2} = \frac{4}{12} = \frac{1}{3}$

line: $y - 2 = \frac{1}{3}(x-0)$ or $\boxed{y = 2 + \frac{1}{3}x}$

(c) On the graph above, sketch the tangent line found in part (b).

above

5. (10 points) A farmer has 900 meters of fencing with which to build two adjacent rectangular pens. One side of the enclosed area borders a river and does not require fencing. (See figure below.) Determine the largest total area that can be enclosed by the two pens.
   You must show your work, use calculus to justify your answer. Your final answer should be a sentence and should include units.



goal: maximize area

$\boxed{\text{way 1}}$: $A = xy$

constraint: $3y + x = 900$

$x = 900 - 3y$

$A(y) = (900-3y)y = 900y - 3y^2$

domain: $(0, \infty)$

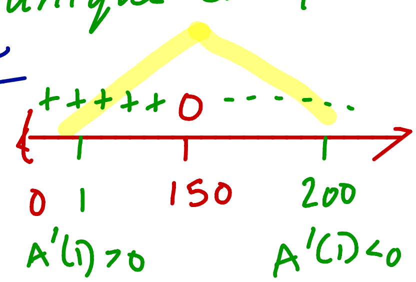
$A'(y) = 900 - 6y = 0, \quad y = 150$

2nd derivative test. $A''(y) = -6 < 0$. So y is a local max.

or

It's an abs. max since it's the unique crit. pt.

1st derivative test



sign of $A'(y)$

So $A(y)$ has a local max at $y = 150$

$\boxed{\text{way 2}}$ $A = 2xy$

constraint: $2x + 3y = 900$

$x = 450 - \frac{3}{2}y$

$A(y) = 2(450 - \frac{3}{2}y)y$

$= 900y - 3y^2$

$$\begin{array}{r} {}^{2}45 \\ 15 \\ \hline 225 \\ 45 \\ \hline 675 \end{array}$$

Final Answer: The maximum area is $(150)(450) = 67{,}500\,m$.

Did you ....

☐ Include a clear justification?

☐ Answer the question?

6. (10 points) Assume the acceleration of an object is given by $a(t) = 12t - 2$ on the interval $[0, 60]$ where t is measured in seconds and $a(t)$ is measured in meters per second

(a) If the initial velocity of the object is 10 meters per second, find the velocity function of the object, $v(t)$.

$v(t) = \int (12t-2)\,dt = 6t^2 - 2t + C$

$10 = v(0) = 6\cdot 0^2 - 2\cdot 0^2 = C$, $C = 10$

$v(t) = 6t^2 - 2t + 10$

(b) If $s = 0$ when $t = 0$, find an equation modeling the position of the object, $s(t)$.

$s(t) = \int v(t)\,dt = \int (6t^2 - 2t + 10)\,dt = 2t^3 - t^2 + 10t + C$

$0 = s(0) = C$. So $C = 0$.

$s(t) = 2t^3 - t^2 + 10t$

(c) Determine the average velocity of the object in the first 3 seconds.

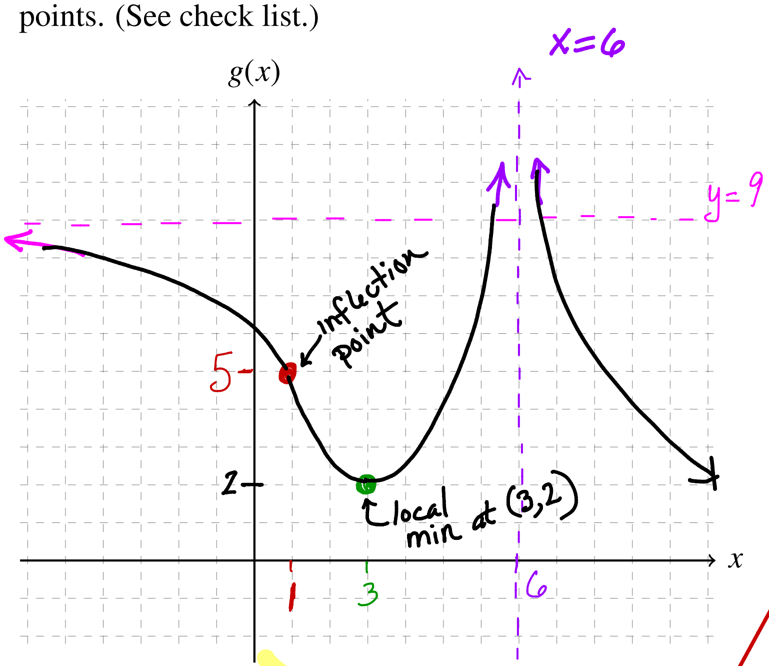
$$\begin{array}{r} {}^{1}27 \\ 2 \\ \hline 54 \end{array}$$

$s(3) = 2\cdot 3^3 - 3^2 + 10\cdot 3 = 84 - 9 = 75$

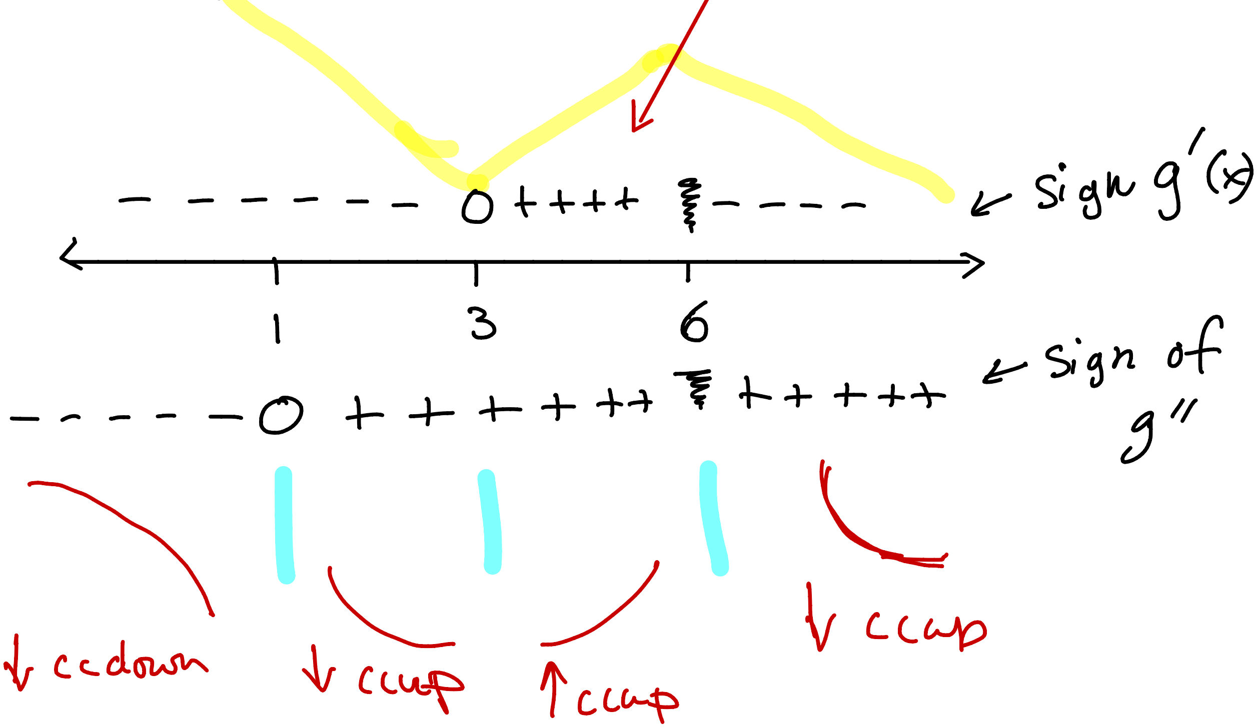
$s(0) = 0$

average velocity $= \dfrac{s(3) - s(0)}{3 - 0} = \dfrac{75}{3} = 25$ m/s

7. (16 points) Use the axes below to sketch a graph of a function $g(x)$ that satisfies **all** of the conditions in the bulleted list. Make sure to label any asymptotes, minimums or maximums, and inflection points. (See check list.)



- $g(x)$ is continuous on its domain $(-\infty, 6) \cup (6, \infty)$.
- $g(1) = 5$, $g''(1) = 0$
- $g(3) = 2$, $g'(3) = 0$
- $\lim_{x \to 6} g(x) = \infty$
- $\lim_{x \to -\infty} g(x) = 9$
- $g'(x) > 0$ on $(3, 6)$ and $g'(x) < 0$ on $(-\infty, 3) \cup (6, \infty)$
- $g''(x) > 0$ on $(1, 6) \cup (6, \infty)$ and $g''(x) < 0$ on $(-\infty, 1)$



Did you ....

☐ label any asymptotes with its equation?

☐ label any maximums or minimums with **local min, local max, absolute min,** or **absolute max**?

☐ label any inflection points with **inflection point**?

8. (12 points) Let $r(t)$ denote that rate at which a thick sheet of volcanic lava cools where r is measured in degrees Farhenheit per day and t is measured in days.

   (a) Write a complete sentence explaining $r(10) = -20$ in the context of the problem. Include units in your answer.

   At the instant 10 days have passed, the sheet of lava is cooling at a rate of 20°F per day.

   (b) Explain why you would expect $r(t) < 0$.

   We are expecting the lava to cool, so its temperature should be decreasing.

   (c) Write a complete sentence explaining $\int_0^{30} r(t)\,dt = -420$ in the context of the problem. Include units in your answer.

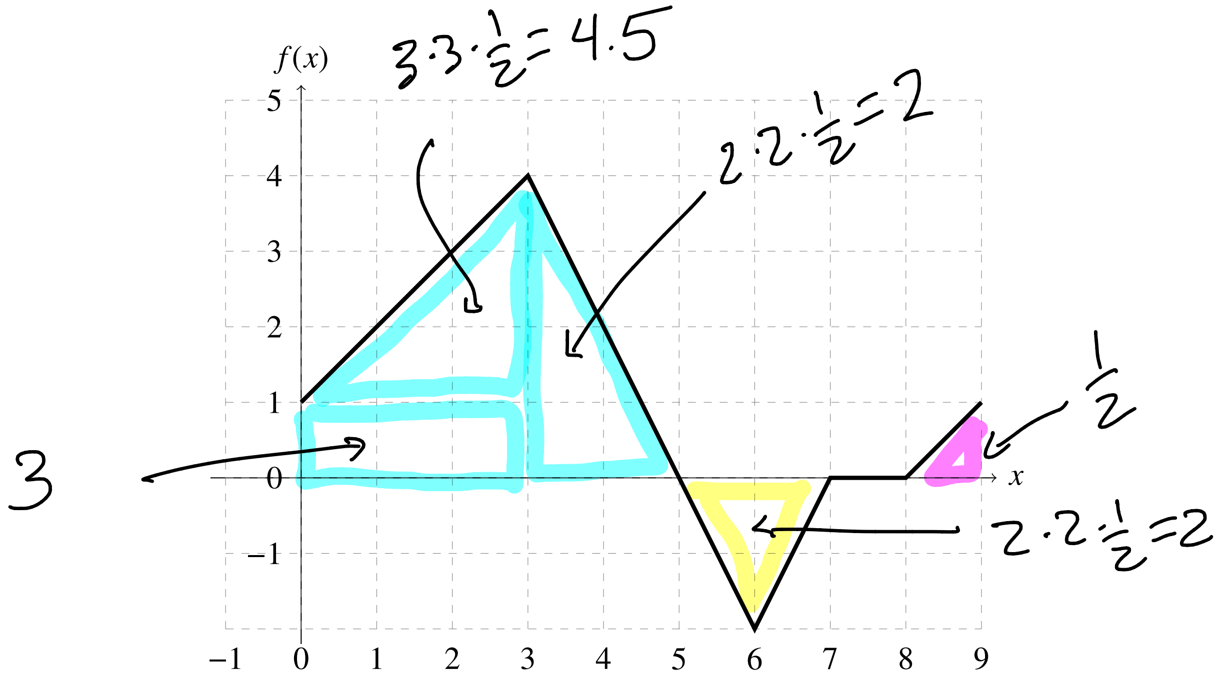
   After 30 days, the lava will be 420°F cooler.

   OR After 30 days, the temperature will have decreased by 420°F.

   (d) Assume the lava began with a temperature of $2080^o F$. Write an expression for the temperature of the lava a year later.

   $$\text{temperture 1 year later} = 2080 + \int_0^{365} r(t)\,dt$$

9. (12 points) Consider the function $f(x)$ with domain $[0, 9]$ graphed below.



(a) What is the value of $f(2)$? 3

(b) What is the value of $f'(2)$? 1

(c) Evaluate $\int_0^9 f(x)\,dx$. $= 3 + 4.5 + 2 - 2 + \frac{1}{2} = 8$

(d) Let $H(x) = \int_0^x f(s)\,ds$. What is the value of $H(4)$? $\int_0^4 f(x)\,dx = 3 + 4.5 + 3 = 10.5$

(e) For $H(x)$ from part **d.**, what is the value of $H'(4)$. $H'(4) = f(4) = 2$

(f) For $H(x)$ from part **d.**, over what interval is $H(x)$ decreasing?

$(5,7)$ or $[5,7]$ or $[5,8]$ or $(5,8)$.

10. (6 points) Let $f(t) = \dfrac{300e^{2t} + 100t}{e^{3t} + 1}$ denote the rate of change of a population of fruit flies where t is measured in days and f is measured in fruit flies per day.

(a) Evaluate $\lim_{t\to\infty} f(t)$. You must show your work to receive full credit.

algebra

$$\lim_{t\to\infty} \frac{\left(300e^{2t} + 100t\right)}{\left(e^{3t} + 1\right)} \frac{\left(\frac{1}{e^{3t}}\right)}{\left(\frac{1}{e^{3t}}\right)} = \lim_{t\to\infty} \frac{\frac{300}{e^t} + \frac{100t}{e^t}}{1 + \frac{1}{e^{3t}}} = 0$$

L'Hopital

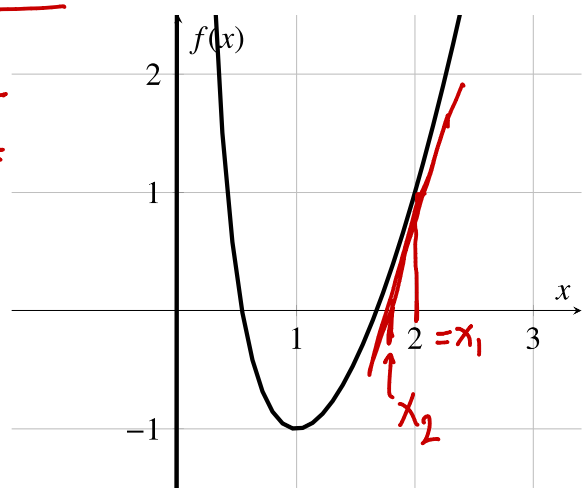
$$\lim_{t\to\infty} \underbrace{\frac{300e^{2t} + 100t}{e^{3t} + 1}}_{\text{form } \frac{\infty}{\infty}} \overset{H}{=} \lim_{t\to\infty} \underbrace{\frac{600e^{2t} + 100}{3e^{3t}}}_{\text{form } \frac{\infty}{\infty}} \overset{H}{=} \lim_{t\to\infty} \frac{1200e^{2t}}{9e^{3t}} = \lim_{t\to\infty} \frac{1200}{9e^t} = 0$$

(b) Explain, using complete sentences, what your calculation in part (a) indicates about the **population** of fruit flies. Your answer should reference the value you obtained in part (a).

Since the rate of change of the population approaches zero as time goes on, the population must stop growing and stabilize at some fixed number of fruitflies.

Extra Credit (5 points) The graph of the function $f(x) = x^2 - \ln(3x)$ is shown.

$$f'(x) = 2x - \frac{1}{3x}\cdot 3 = 2x - \frac{1}{x}$$



a. Suppose Newton's method is used to find an approximate solution to $f(x) = 0$ from an initial guess of $x_1 = 2$. Sketch on the graph how the next approximation x_2 will be found, labeling its location on the x-axis.

b. For $x_1 = 2$, give a formula for x_2. You do not need to simplify, but your answer should be in a form where a calculator would compute a numerical value.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{4 - \ln(6)}{3.5}$$

$$f(2) = 2^2 - \ln(6) = 4 - \ln(6)$$

$$f(2) = 2\cdot 2 - \frac{1}{2} = 3.5$$