Math F251 Midterm 2 Fall 2022

Name: Solutions

Rules:

You have ~~60~~ 90 minutes to complete the exam.
Partial credit will be awarded, but you must show your work.
You may have a single handwritten 3×5 notecard.
Calculators are not allowed.
Place a box around your FINAL ANSWER to each question where appropriate.
Turn off anything that might go beep during the exam.
Good luck!

Problem	Possible	Score
1	10	
2	10	
3	10	
4	10	
5	18	
6	10	
7	10	
8	10	
9	12	
Extra Credit	5	
Total	100	

1. (10 points) Evaluate the following limits. **You must show your work and justify your answer to earn full credit.** If you apply L'Hopital's Rule, you should indicate this, by writing $\overset{H}{=}$ or $\overset{L'H}{=}$ or some other clear indication.

   (a) $\lim_{x\to\infty} \frac{\sqrt{x^2+4}}{5x+4}\left(\frac{\frac{1}{x}}{\frac{1}{x}}\right) = \lim_{x\to\infty} \frac{\sqrt{\frac{x^2+4}{x^2}}}{5+\frac{4}{x}} = \lim_{x\to\infty} \frac{\sqrt{1+\frac{4}{x^2}}}{5+\frac{4}{x}} = \frac{1}{5}$

   $1 \quad 0 \quad 5 \quad 0$

   (b) $\underbrace{\lim_{x\to 0} \frac{x^2}{3-3\cos(x)}}_{\text{form } \frac{0}{0}} \overset{H}{=} \lim_{x\to 0} \underbrace{\frac{2x}{3\sin(x)}}_{\text{form } \frac{0}{0}} \overset{H}{=} \lim_{x\to 0} \frac{2}{3\cos(x)} = \frac{2}{3}$

   using $\cos(0)=1$

2. (10 points)

   (a) Find the linear approximation (also known as the linearization) of the function $f(x) = \sqrt{x}$ when $a = 1$.

   $f(x) = x^{1/2}$; $f(1)=1$

   $f'(x) = \frac{1}{2}x^{-1/2}$; $f'(1) = \frac{1}{2} = 0.5$

   point $(1,1)$

   slope $= m = 0.5$
   $= \frac{1}{2}$

   $y-1 = 0.5(x-1)$

   $y = 1+0.5(x-1)$

   ANSWER:

   $L(x) = 1+0.5(x-1)$ OR

   $L(x) = 1+\frac{1}{2}(x-1)$

   (b) Use the linear approximation from part (a) to estimate $\sqrt{1.05}$. Your answer must be in the form of a simplified decimal or an exact fraction.

   decimals

   $\sqrt{1.05} \approx L(1.05) = 1+0.5(1.05-1) = 1+(0.5)(0.05)$

   $= 1+0.025 = 1.025$

   fractions

   $\sqrt{1.05} \approx L(1.05) = 1+\frac{1}{2}(1.05-1) = 1+\frac{1}{2}(0.05) = 1+\frac{1}{2}\left(\frac{5}{100}\right) = 1+\frac{5}{200}$

   $= \frac{205}{200} = \frac{41}{40}$

3. (10 points) The formula for the volume, V, of a cone in terms of its radius r and height h is $V = \frac{1}{3}\pi r^2 h$. If the volume of the cone remains **constant** and the radius of the cone is increasing at a rate of 2 cm/s, determine the rate of change of the height of the cone at the instant the radius is 10 cm and the height is 20 cm. **Interpret your answer using a compete sentence.**

$$\boxed{\begin{array}{l} V = \frac{1}{3}\pi r^2 h \\ \frac{dV}{dt} = 0 \\ \frac{dr}{dt} = 2 \text{ cm/s} \\ \text{Find } \frac{dh}{dt} \text{ when } r=10,\ h=20 \end{array}}$$

Set up

$$\frac{dV}{dt} = \left(\frac{1}{3}\pi\right)\left(2r\frac{dr}{dt}h + r^2\frac{dh}{dt}\right)$$

Take derivative.
- wrt time t
- use prod. rule

$$0 = \left(\frac{\pi}{3}\right)\left(2(10)(2)(20) + 10^2\frac{dh}{dt}\right)$$

Plug in

$$-100\frac{dh}{dt} = 800$$

$$\frac{dh}{dt} = -8 \text{ cm/s}$$

Solve for $\frac{dh}{dt}$

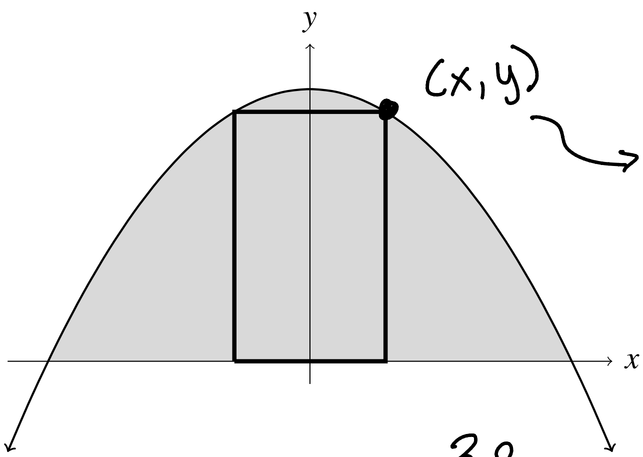
The height is decreasing at a rate of 8 cm/s.

answer

get direction correct

get units correct

4. (10 points) Determine the maximum area of a rectangle with base on the x-axis inscribed between the parabola $y = 12 - x^2$ and the x-axis (See figure below.) Note: Your solution must use Calculus to **justify** that your answer is correct.



goal: maximize area

$y = 12 - x^2$

A = area of rectangle $= 2xy$

answer: Maximum area is 32

Need Area, A, as a function of 1 variable.
So substitute in for y.

There are many correct answers, here.

$A(x) = 2x(12-x^2) = 2(12x - x^3)$; domain $(0, \sqrt{12})$

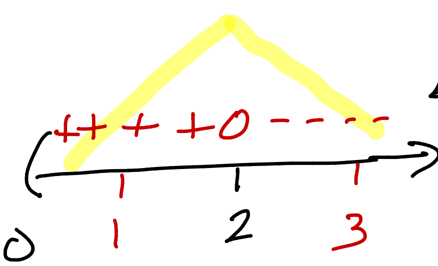
Find critical points:

$A'(x) = 2(12-3x^2) = 0$. So $12 - 3x^2 = 0$ or $x^2 = 4$, $x = \pm 2$

Only $x = +2$ is in my domain.

Check it's a maximum!

First Derivative Test:



sign of $A'(x)$

$A'(1) = (2)(12-3) > 0$
$A'(3) = (2)(12-27) < 0$

So $A(x)$ has a maximum at $x = 2$.

Plug $x=2$ into $A(x)$ to find the maximum area:

$A(2) = 2 \cdot 2(12 - 2^2) = 4(8) = 32$

5. (18 points) Use the information below to answer questions about the function $f(x)$. You must show your work to earn full credit.

$$f(x) = \frac{x}{e^x}, \quad f'(x) = \frac{-(x-1)}{e^x}, \quad f''(x) = \frac{x-2}{e^x}.$$

(a) Determine the intervals on which $f(x)$ is increasing/decreasing.

$f' = 0$ when $x = 1$, f' never undefined

+ + + + 0 - - - - ← Sign of f'

0 1 2

$f'(0) = \frac{(-)(-)}{+} > 0$

$f'(2) = \frac{(-)(+)}{-} < 0$

ANSWER:

$f(x)$ is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$

(b) Find the x-values that correspond to any local maximums or local minimums of $f(x)$.

f has a maximum at $x = 1$

f has no local minimum.

(c) Find the intervals on which $f(x)$ is concave up and concave down.

$f''(x) = 0$ when $x = 2$

f'' is never undefined

- - - - 0 + + + ← Sign of f''

0 2 4

$f''(0) = \frac{-}{+} < 0$, $f''(4) = \frac{+}{+} > 0$

Answer:

$f(x)$ is ccup on $(2, \infty)$ and ccdown on $(-\infty, 2)$.

(d) Find the x-values of any inflection points of $f(x)$. If there aren't any, you must explicitly state this and justify your answer.

f has an inflection point at $x = 2$.

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Note the function and its derivatives are:

$$f(x) = \frac{x}{e^x}, \quad f'(x) = \frac{-(x-1)}{e^x}, \quad f''(x) = \frac{x-2}{e^x}.$$

(e) Give the equation of any horizontal asymptotes of $f(x)$ or state that none exist. Justify your answer using Calculus.

For horizontal asymptotes, check limit of $f(x)$ as $x \to \pm\infty$.

$$\lim_{x\to+\infty} \frac{x}{e^x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0$$

form $\frac{\infty}{\infty}$.

So $y=0$ is a horizontal asymptote

$$\lim_{x\to-\infty} \frac{x}{e^x} = \lim_{x\to\infty} -xe^x = -\infty.$$

So $y=0$ is the only horizontal asymptote.

(f) Give the equation of any vertical asymptotes of $f(x)$ or state that none exist. Justify your answer using using Calculus.

No vertical asymptotes.

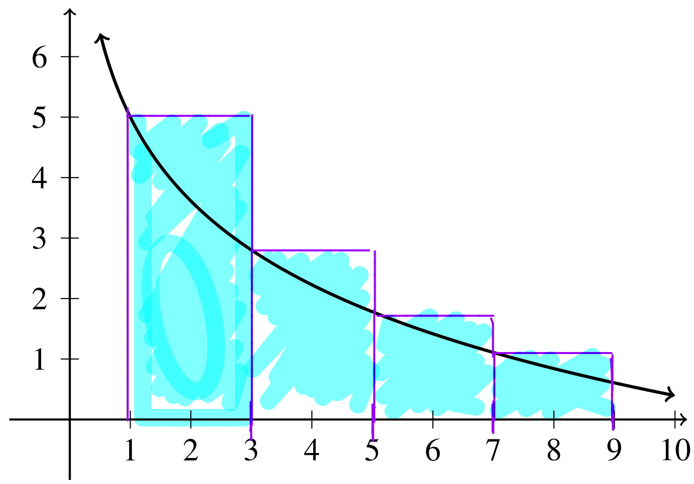
The denominator e^x is never zero.

So there is no x-value a so that

$$\lim_{x\to a^{+/-}} \frac{x}{e^x} = \pm\infty$$

6. (10 points) The function $f(x) = 5 - 2\ln(x)$ is graphed below. We want to estimate the area under the curve $f(x) = \ln(x)$ on the interval $[1, 9]$ using L_4. (That is, we want to use 4 approximating rectangles and left-hand end points.)

   (a) Sketch the four approximating rectangles on the graph.



   (b) Do a calculation to estimate the area under the curve using L_4 (that is, use 4 approximating rectangles and left-hand end points) and simplify your answer. Note: You are obviously not expected to compute things like $\ln(4)$. It is acceptable to have numbers like this in your final answer.

$$A \approx 2\left(f(1) + f(3) + f(5) + f(7)\right)$$

$$= 2\left(5 - 2\ln(1) + 5 - 2\ln(3) + 5 - 2\ln(5) + 5 - 2\ln(7)\right)$$

$$= 40 - 4\left(\ln(3) + \ln(5) + \ln(7)\right)$$

This is an acceptable final answer.

7. (10 points) Evaluate the indefinite integrals below.

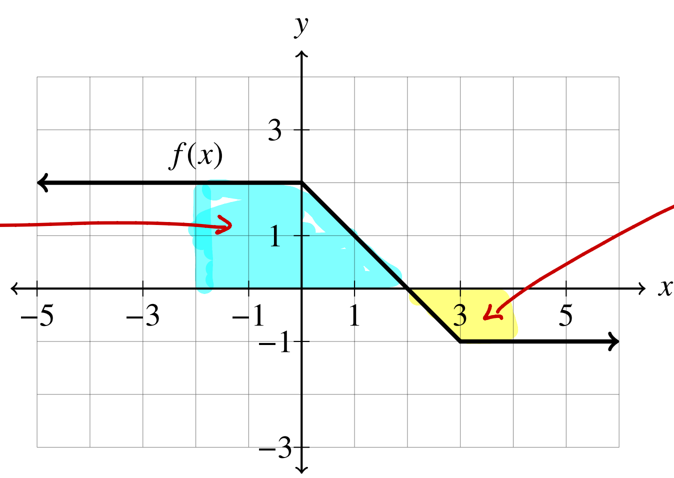
   (a) $\int (4\sin(x) + x^5 + x^{-1} + 10)\,dx$

   $= -4\cos(x) + \frac{1}{6}x^6 + \ln|x| + 10x + C$

   (b) $\int \frac{1+x^3}{x^2}\,dx = \int (x^{-2} + x)\,dx = -x^{-1} + \frac{1}{2}x^2 + C$

8. (10 points) Evaluate the definite integrals below using the graph of $f(x)$ and properties of definite integrals. Show your work.

area = 6



area = 1.5

   (a) $\int_{-2}^{4} f(x)\,dx = 6 - 1.5 = 4.5$

   (b) $\int_{-2}^{4} (5f(x) + 3)\,dx = 5\int_{-2}^{4} f(x)\,dx + \int_{-2}^{4} 3\,dx$

   $= 5(4.5) + 3(6) = 40.5$

height base

$$\begin{array}{r} 4.5 \\ \times\ 5 \\ \hline 22.5 \\ 18 \\ \hline 0.5 \end{array}$$

9. (12 points) Sketch the graph of a function $f(x)$ that satisfies all of the given conditions. Clearly label any important points on the x-axis, draw any asymptotes clearly as dashed lines, and label any asymptotes with their equations.

   Something bad happens when $x=1$!

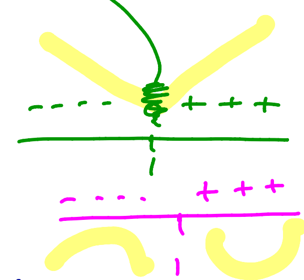
   (a) The domain of $f(x)$ is $(-\infty, \infty)$.

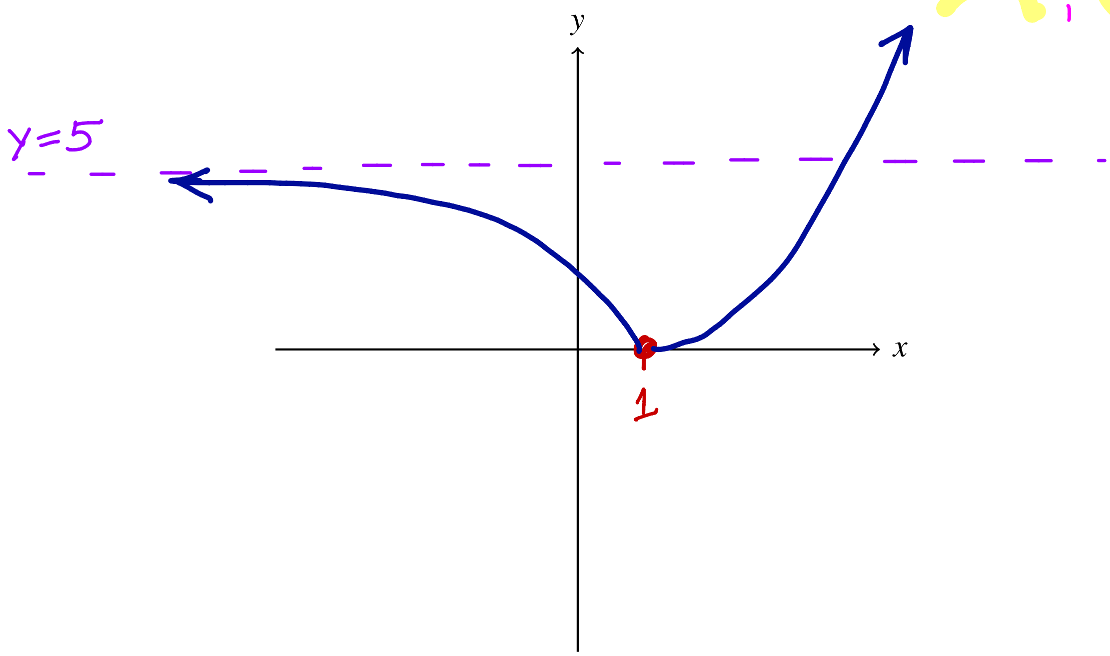
   (b) $f(1) = 0$ and $f'(1)$ is undefined

   (c) $\lim_{x \to -\infty} f(x) = 5$ h.a. at $y=5$

   (d) $f'(x) < 0$ on the interval $(-\infty, 1)$; $f'(x) > 0$ on the interval $(1, \infty)$

   (e) $f''(x) < 0$ on the interval $(-\infty, 1)$; $f''(x) > 0$ on the interval $(1, \infty)$





Extra Credit: Suppose $C(t)$ models the position of a car and $B(t)$ models the position of a bike over the same time interval, $[0, 2]$, where C and B are measured in miles and t in hours.
(3 pts) Translate the following sentence into the language of Calculus: "The car goes faster than the bike but the bike accelerates faster than the car." (i.e. rewrite the sentence using derivatives in some form.)
(2pts) Construct a pair of functions $C(t)$ and $B(t)$ satisfying the properties described in the sentence on the interval $[0, 2]$. (Note, your functions do not have to be realistic...)

translation: $C'(t) > B'(t)$ but $B''(t) > C''(t)$

reverse engineer an example: Make $B''(t) = 5$ and $C''(t) = 0$

So $B'(t) = 5t + C$ and $C'(t) = D$. Pick $C = 0$ and $D = 100$. ← Note: Irresponsibly high speed!

So $B'(t) = 5t$ and $C'(t) = 100$.

ANSWER: $B(t) = \frac{5}{2}t^2$, $C(t) = 100t$.