Math F251 — Final Exam — Spring 2019

Name: SOLUTIONS

Section: ☐ F01 (Rhodes) ☐ F02 (Bueler)

Rules:

You have 2 hours to complete the exam.

Partial credit will be awarded, but you must show your work.

No calculators, books, notes, or other aids are permitted.

Circle your final answer to each question where appropriate.

If you need extra space, you can use the back sides of the pages. (Clearly label any work you want graded.)

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	10	
2	10	
3	15	
4	15	
5	10	
6	5	
7	15	
8	15	
9	15	
10	10	
11	10	
Extra Credit	3	
Total	130	

Math 251: Final Exam

1. (10 points)

Find an equation of the tangent line to the curve at $x = e$: $y = x^2 \ln x$

$$y' = 2x \ln x + x^2 \cdot \frac{1}{x} = 2x \ln x + x$$

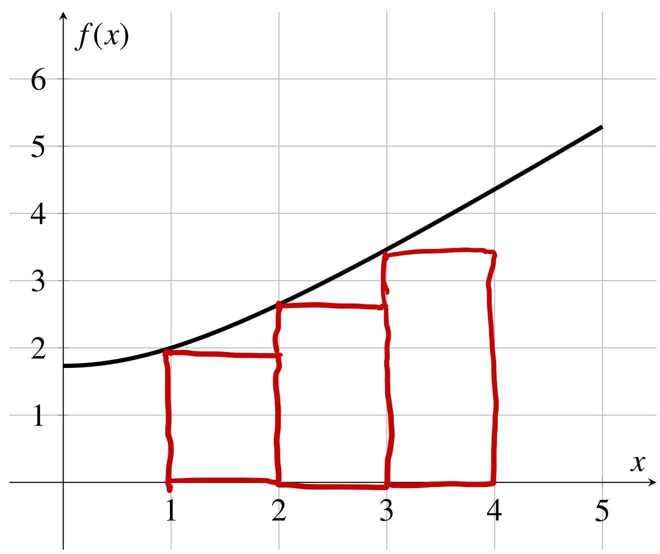
$$m = y'|_e = 2 \cdot e \cdot 1 + e = 3e$$

$$y_0 = y|_e = e^2 \cdot 1 = e^2$$

$$y - e^2 = 3e(x - e)$$

2. (10 points)

The graph of the function $f(x) = \sqrt{x^2 + 3}$ is shown.



a. On the graph sketch 3 rectangles, using left endpoints, that would be used to approximate

$$\int_1^4 \sqrt{x^2 + 3}\, dx.$$

b. Compute the approximation in part **(a)**. You do not need to simplify, but your answer should be in a form where a calculator would compute a numerical value.

$$\Delta t = \frac{4-1}{3} = 1$$

$$L_3 = \sqrt{1^2+3} \cdot 1 + \sqrt{2^2+3} \cdot 1 + \sqrt{3^2+3} \cdot 1$$

$$= \sqrt{4} + \sqrt{7} + \sqrt{12}$$

$$= 2 + \sqrt{7} + 2\sqrt{3}$$

Math 251: Final Exam

3. (15 points)

Find the following limits.

a. $\lim_{t\to 5} \frac{\frac{1}{t}-\frac{1}{5}}{t-5}$ $= \lim_{t\to 5} \frac{\frac{5-t}{5t}}{t-5} = \lim_{t\to 5} \frac{\cancel{5-t}^{\,-1}}{5t(\cancel{t-5})}$

$= \lim_{t\to 5} \frac{-1}{5t} = \frac{-1}{25}$

b. $\lim_{x\to\infty} \frac{2x^2+3}{3x^2-5x}$ $= \lim_{x\to\infty} \frac{2+\frac{3}{x^2}}{3-\frac{5}{x}} = \frac{2+0}{3-0}$

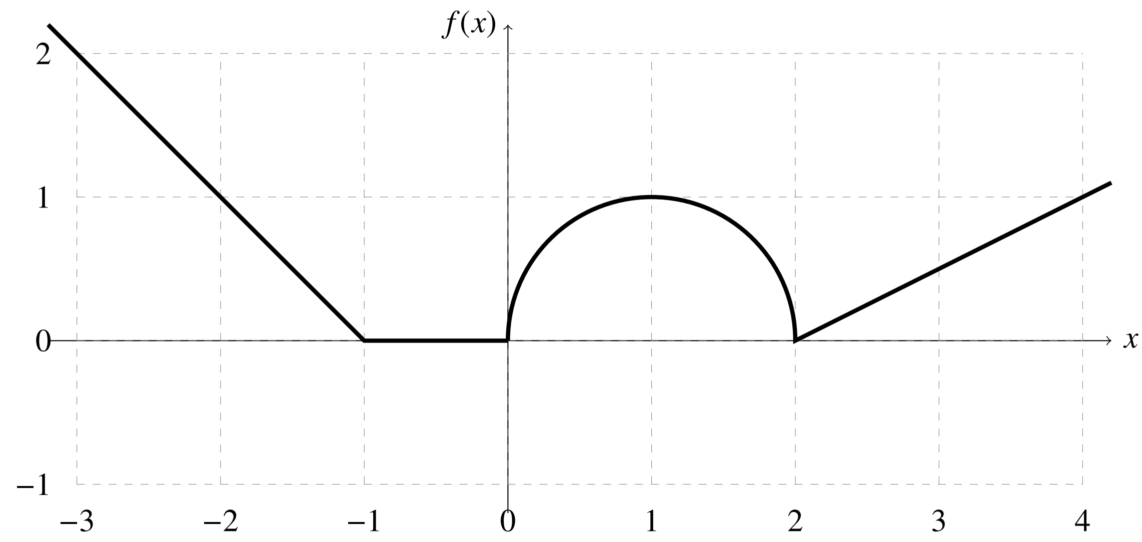
$= \frac{2}{3}$ [L'Hopital's twice is fine too]

c. $\lim_{\theta\to 0} \frac{\theta^2}{1-\cos\theta}$ $\underset{L'H}{\overset{\frac{0}{0}}{=}} \lim_{\theta\to 0} \frac{2\theta}{+\sin\theta} \underset{L'H}{\overset{\frac{0}{0}}{=}} \lim_{\theta\to 0} \frac{2}{\cos\theta}$

$= \frac{2}{1} = 2$

4. (15 points)

Consider the function $f(x)$ graphed below. Between $x = 0$ and 2, the graph is of a semicircle of radius 1.



a. At what x values, if any, does $f'(x)$ not exist?

f' d.n.e. at $x = -1, 0, 2$

b. What is the value of $f'(-2)$?

$f'(-2) = (\text{slope}) = -1$

c. Evaluate $\int_{-1}^{4} f(x)\,dx$.

$$\int_{-1}^{4} f(x)dx = 0 + \frac{1}{2}\pi + 1 = \frac{\pi}{2} + 1$$

d. Let $g(x) = \int_{1}^{x} f(s)\,ds$. What is the value of $g(0)$?

$$g(0) = \int_{1}^{0} f(s)ds = -\int_{0}^{1} f(s)ds = -\frac{1}{4}\pi$$

e. For $g(x)$ from part **d.**, what is the value of $g'(4)$.

$$g'(4) \underset{\text{FTCI}}{=} f(4) = 1$$

5. (10 points)

Differentiate the following functions.

a. $y = \cos\left(e^{\sin x}\right)$

$$y' = -\sin\left(e^{\sin x}\right) \cdot e^{\sin x} \cdot \cos x$$

b. $g(t) = \dfrac{1 - 2t^5}{1 + \tan t}$

$$g'(t) = \frac{-10t^4(1+\tan t) - (1-2t^5)(\sec^2 t)}{(1+\tan t)^2}$$

6. (5 points)

Find $\dfrac{dy}{dx}$ by implicit differentiation: $\qquad e^y + \dfrac{1}{x} = xy + 2y.$

$$e^y \frac{dy}{dx} - x^{-2} = 1 \cdot y + x \cdot \frac{dy}{dx} + 2\frac{dy}{dx}$$

$$(e^y - x - 2)\frac{dy}{dx} = x^{-2} + y$$

$$\frac{dy}{dx} = \frac{x^{-2} + y}{e^y - x - 2}$$

7. (15 points)

A particle moves so that its velocity (in m/sec) at time t sec is

$$v(t) = t^2 + 7.$$

a. What is the average change in the velocity of the particle from time $t = 2$ to $t = 3$? Simplify, and give units.

$$\frac{v(3) - v(2)}{3-2} = \frac{(3^2+\cancel{7}) - (2^2+\cancel{7})}{1} = 5 \frac{m}{s^2}$$

b. Using the limit definition of the derivative, compute $v'(2)$. (No credit will be given for using a different method to compute the derivative.)

$$v'(2) = \lim_{h\to 0} \frac{v(2+h) - v(h)}{h}$$

$$= \lim_{h\to 0} \frac{(2+h)^2+\cancel{7} - (2^2+\cancel{7})}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{4} + 4h + h^2 - \cancel{4}}{h}$$

$$= \lim_{h\to 0} 4 + h = 4 + 0 = 4 \frac{m}{s^2}$$

8. (15 points)

Evaluate the integrals. For full credit, include a constant of integration whenever one would be justified.

a. $\int \sin^5(x)\cos x\, dx =$

$$\left[\begin{array}{l} u=\sin x \\ du=\cos x\,dx \end{array}\right]$$

$$\int u^5\,du = \frac{1}{6}u^6 + C$$

$$= \frac{1}{6}(\sin x)^6 + C$$

b. $\int_1^3 2e^x + \frac{1}{x}\, dx =$ $2e^x + \ln|x| \Big]_1^3$

$$= (2e^3 + \ln 3) - (2e^1 + \cancel{\ln 1}^{\,0})$$

$$= 2e^3 + \ln 3 - 2e$$

c. $\int \sqrt{x}\left(x^2 - x^{1/4} + \pi^2\right) dx =$ $\int x^{5/2} - x^{3/4} + \pi^2 x^{1/2}\, dx$

$$= \frac{2}{7}x^{7/2} - \frac{4}{7}x^{7/4} + \pi^2 \cdot \frac{2}{3}x^{3/2} + C$$

9. (15 points)

Consider the following function:

$$f(x) = \frac{x}{x^2 - 1}$$

a. What is the domain of f?

$x \neq -1, +1$ (or: $(-\infty,-1) \cup (-1,1) \cup (1,\infty)$)

b. Find all critical numbers of f, if any.

$$f'(x) = \frac{1 \cdot (x^2-1) - x(2x)}{(x^2-1)^2} = \frac{-1-x^2}{(x^2-1)^2} = -\frac{x^2+1}{(x^2-1)^2}$$

$f'(x) = 0$ has no solutions $\therefore$ no critical points

c. Determine the intervals on which f is increasing or decreasing.

$f'(x) < 0$ everywhere it is defined

$\therefore$ decreasing on $(-\infty,-1) \cup (-1,1) \cup (1,\infty)$

increasing nowhere

d. Find all asymptotes of f, both vertical and horizontal. (Identify each asymptote as either vertical or horizontal.)

Vertical: $x = -1$ and $x = +1$

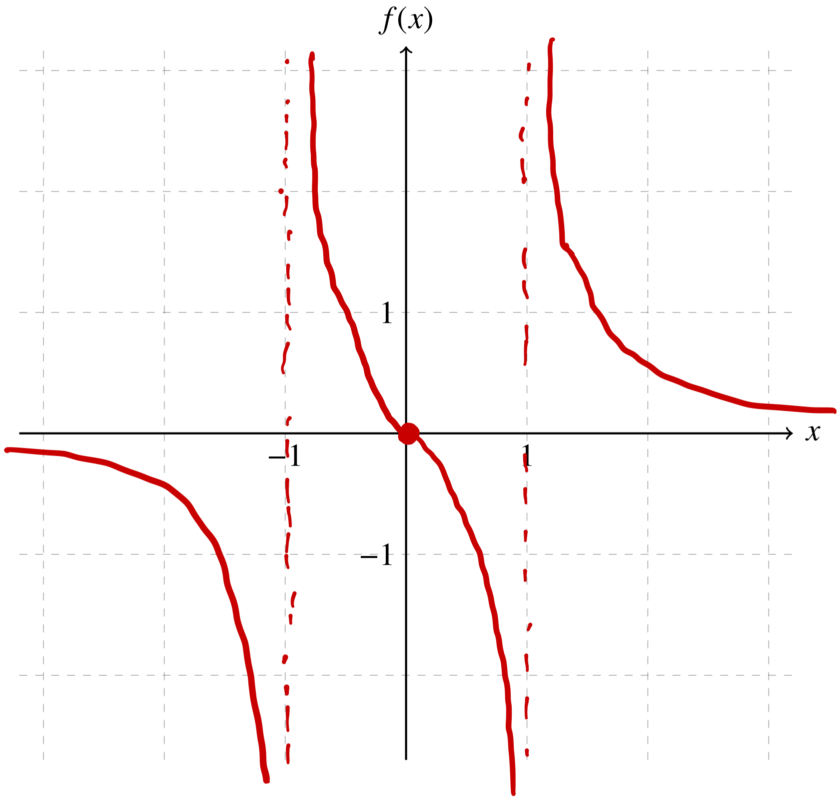
horizontal: $y = 0$

$$\left[\lim_{x\to\infty} \frac{x}{x^2-1} \overset{LH}{\underset{\frac{\infty}{\infty}}{=}} \lim_{t\to\infty} \frac{1}{2x} = 0\right]$$

Math 251: Final Exam

Problem 9 continued....

e. Sketch the graph on the axes:



① note $f(x)$ is odd

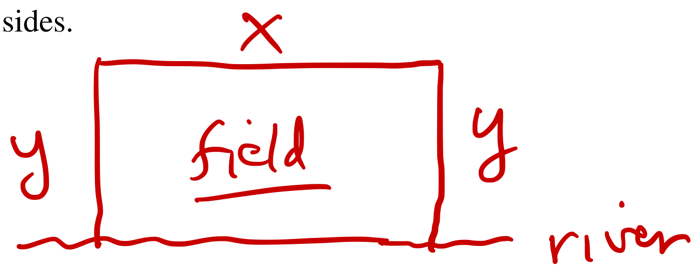
② note $f(x) > 0$ if $x > 1$

③ $f(0) = 0$

10. (10 points)

A farmer has 400 meters of fencing and wants to fence off a rectangular field that borders a straight river. No fencing is needed along the river, which forms one side of the rectangle. What are the dimensions of the field that has the largest area? Indicate units in your answer.

a. Draw a sketch and choose labels for the sides.



b. Solve the problem.

$$\left.\begin{aligned} x + 2y &= 400 \\ A &= xy \end{aligned}\right\} \therefore A(y) = (400 - 2y)y = 400y - 2y^2$$

$$A'(y) = 400 - 4y = 0 \quad \therefore \quad y = 100$$

$x = 200$ m, $y = 100$ m

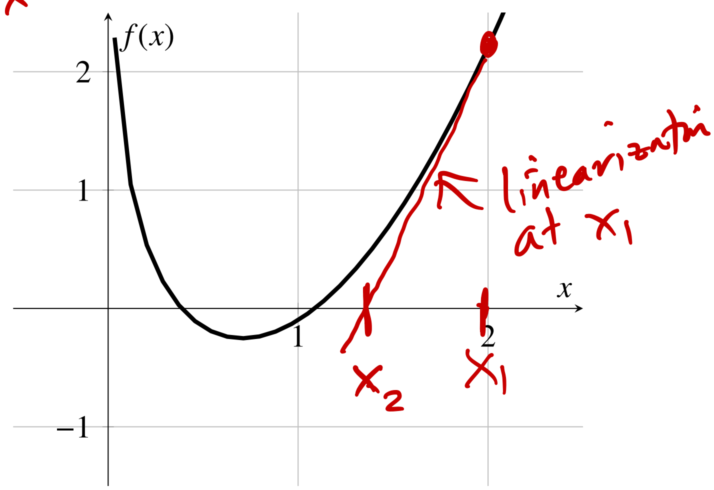
y	A
0	0
100	$200 \cdot 100 = 20{,}000$ m^2
200	0

Math 251: Final Exam

$f'(x) = 2x - \frac{1}{3x}\cdot 3 = 2x - \frac{1}{x}$

11. (10 points)

The graph of the function $f(x) = x^2 - \ln(3x)$ is shown.



a. Suppose Newton's method is used to find an approximate solution to $f(x) = 0$ from an initial guess of $x_1 = 2$. Sketch on the graph how the next approximation x_2 will be found, labeling its location on the x-axis.

b. For $x_1 = 2$, give a formula for x_2. You do not need to simplify, but your answer should be in a form where a calculator would compute a numerical value.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1^2 - \ln(3x_1)}{2x_1 - \frac{1}{x_1}}$$

$$= 2 - \frac{4 - \ln 6}{4 - \frac{1}{2}} = 2 - \frac{2}{7}(4 - \ln 6)$$

c. What value of x_1 might you use if you wanted to find the **smaller** solution of $f(x) = 0$?

$x_1 = 0.3$ perhaps [any x_1 in $[0, 0.7]$]

Extra Credit. (3 points)

Compute the following integral by interpreting it as an area:

$$\int_0^4 \sqrt{4 - (x-2)^2}\, dx = \frac{1}{2}\pi \cdot 2^2 = 2\pi$$

$y = \sqrt{4 - (x-2)^2}$

$y^2 = 4 - (x-2)^2$

$(x-2)^2 + y^2 = 2^2$