Math F251 Midterm 2 Spring 2020

Name: Solutions

Section: ☐ F01 (Faudree)
☐ F02 (Bueler)
☐ UX1 (Van Spronsen)

All students must affirm the following statements by initialing in the blanks provided. Students using their own paper must write out the statements in full.

✓ I will not seek or accept help from anyone.

✓ I will not use a calculator, books, notes, the internet or other aids.

✓ I understand that answers without work will not be awarded credit.

Good luck!

Problem	Possible	Score
1	10	
2	10	
3	10	
4	8	
5	12	
6	6	
7	12	
8	10	
9	10	
10	12	
Total	100	

1. (10 points)

A table of values for $f(x)$, $g(x)$, $f'(x)$ and $g'(x)$ is given.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	1	2	8
2	4	3	2	4
3	5	2	1	6

a. If $h(x) = x^2 f(x) - g(x)$, find $h'(3)$.

$$h'(x) = x^2 f'(x) + 2xf(x) - g'(x)$$
$$h'(3) = (3)^2 \cdot 2 + 2\cdot 3\cdot 5 - 6 = 18 + 30 - 6 = \boxed{42}$$

b. If $h(x) = f(g(x))$, find $h'(1)$.

$$h'(x) = f'(g(x)) \cdot g'(x)$$
$$h'(1) = f'(g(1)) \cdot g'(1) = f'(2) \cdot 8$$
$$= 3 \cdot 8 = 24$$

2. (10 points)

A particle moves on a vertical line so that its coordinate y at time t is $y = t^4 - 3t^2 + 2$, where $t \geq 0$.

a. What is the initial position of the particle?

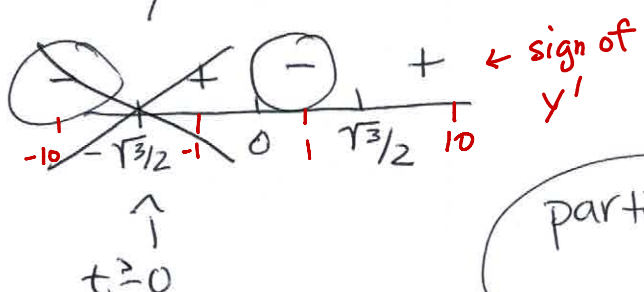
when $t = 0$, $y = 2$

b. When is the particle moving downward?

$$y' = 4t^3 - 6t$$
$$y' = 2t(2t^2 - 3)$$

$y' = 0$ when $t = 0$ and $2t^2 - 3 = 0$

$$2t^2 = 3$$
$$t^2 = 3/2$$
$$t = \pm\sqrt{3/2}$$



particle is moving downward when t is in the interval $(0, \sqrt{3/2})$

3. (10 points)

On March 21, the Alaska Department of Health and Social Services finds 21 Alaskans are infected with a new virus. By March 31, the number of Alaskans infected has risen to 133. Assume that the number of people infected grows at a rate proportional to the size of the infected population.

a. Write an equation that says that the number of people infected grows at a rate proportional to the size of the infected population.

$$\frac{dN}{dt} = kN$$

b. Assuming the growth rate continues, with no mitigating factors, find an expression for the number, N, of Alaskans infected over time t in days.

$$N(t) = 21e^{kt}$$

$$133 = 21e^{k(10)}$$

$$\frac{133}{21} = e^{10k}$$

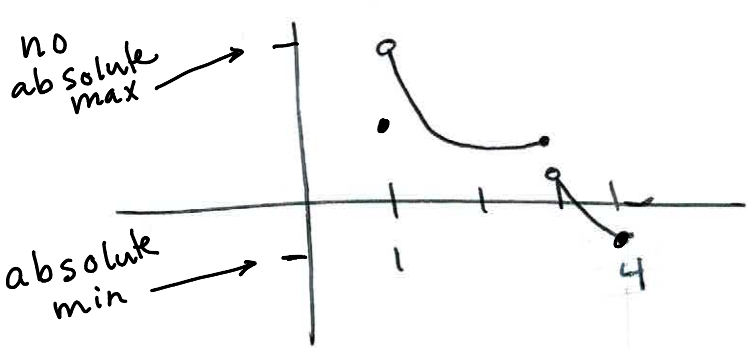
$$\ln(133/21) = 10k$$

$$\frac{\ln(133/21)}{10} = k$$

$$N = 21e^{\left(\frac{\ln(133/21)}{10}\right)t}$$

4. (8 points)

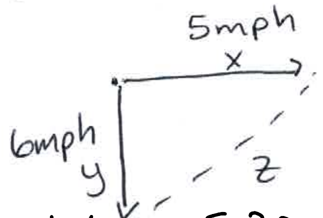
Sketch a graph f with domain $[1, 4]$ such that f has an absolute minimum but no absolute maximum.



* there are lots of options here.

5. (12 points)

A ship passes a lighthouse at 3:30pm, sailing to the east at 5 mph, while another ship sailing due south at 6 mph passes the same point half an hour later. How fast will the distance between the ships be increasing at 5:30pm?



Want: dz/dt @ 5:30 pm.

$$x^2 + y^2 = z^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

at 5:30pm

$x = 10$ $\frac{dx}{dt} = 5$

$y = 9$ $\frac{dy}{dt} = 6$

$$\frac{dz}{dt} = \frac{2x\frac{dx}{dt} + 2y\frac{dy}{dt}}{2z}$$

$10^2 + 9^2 = z^2$

$\sqrt{181} = z$

$$\frac{dz}{dt} = \frac{\cancel{2}(10)\cdot 5 + \cancel{2}(9)(6)}{\cancel{2}\sqrt{181}}$$

$$\frac{dz}{dt} = \frac{104}{\sqrt{181}} \text{ mph}$$

$$\begin{array}{r} 50 \\ +54 \\ \hline 104 \end{array}$$

6. (6 points)

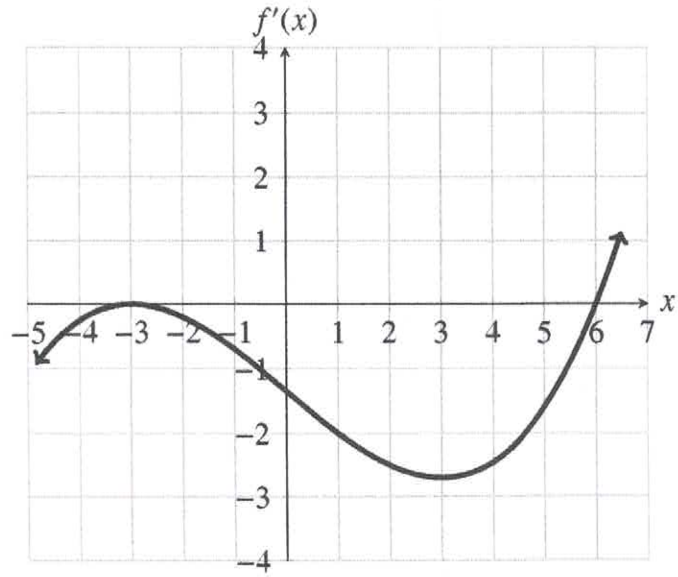
Does the graph of the function $f(x) = \dfrac{3\ln x}{1-x}$ have a vertical asymptote at $x = 1$? Justify your answer using an appropriate limit.

$$\lim_{x\to 1} \underbrace{\frac{3\ln x}{1-x}}_{\text{form } \frac{0}{0}} \overset{H}{=} \lim_{x\to 1} \frac{\frac{3}{x}}{-1} = -3$$

No, f(x) does not have a vertical asymptote at $x=1$ because the limit as $x\to 1$ is not $\pm\infty$.

7. (12 points)

The graph of the ***derivative*** f' of a continuous function f is shown.



a. Determine the critical points of $f(x)$.

where graph $=0$, so

$x = -3, 6$

b. At what values of x, does f have a local maximum? Local minimum? Explain your answer.

- Since f' is negative both before and after $x=-3$, this is neither a local max OR min
- at $x=6$, f' changes from negative to positive so there is a local min at $x=6$.

c. On what intervals is f concave upward? Concave downward? Use interval notation.

where f' is increasing, f is concave up, which is $(-\infty,-3) \cup (3,\infty)$

where f' is decreasing, f is concave down, which is $(-3,3)$

8. (10 points)

A function and its first and second derivatives are given below.

$$f(x) = x^{5/3} - 5x^{2/3}, \qquad f'(x) = \frac{5x - 10}{3x^{1/3}}, \qquad f''(x) = \frac{10x + 10}{9x^{4/3}}$$

a. Find the intervals of increase and decrease, and identify the locations of any local maximum or minimum values.

$f'(x) = 0$ when $5x - 10 = 0$, $5x = 10$, $x = 2$

and undefined when $x = 0$

sign of f': $+$ on $(-\infty, 0)$, $-$ on $(0, 2)$, $+$ on $(2, \infty)$ (test points -1, 1, 3)

f is increasing on the interval $(-\infty, 0) \cup (2, \infty)$

and decreasing on the interval $(0, 2)$

f has a local max at $x = 0$ and a local min at $x = 2$.

b. Find the intervals of concavity and the x-values of any inflection points.

$f''(x) = 0$ when $x = -1$

and undefined at $x = 0$

Sign of f'': $-$ on $(-\infty, -1)$, $+$ on $(-1, 0)$, $+$ on $(0, \infty)$ (test points -2, $-\frac{1}{2}$, 1)

f is concave up on $(-1, \infty)$

and concave down on $(-\infty, -1)$

f has an inflection point at $x = -1$.

9. (10 points)

Sketch a graph that satisfies all of the conditions:

domain $f = (-\infty, \infty)$, critical pt and local min

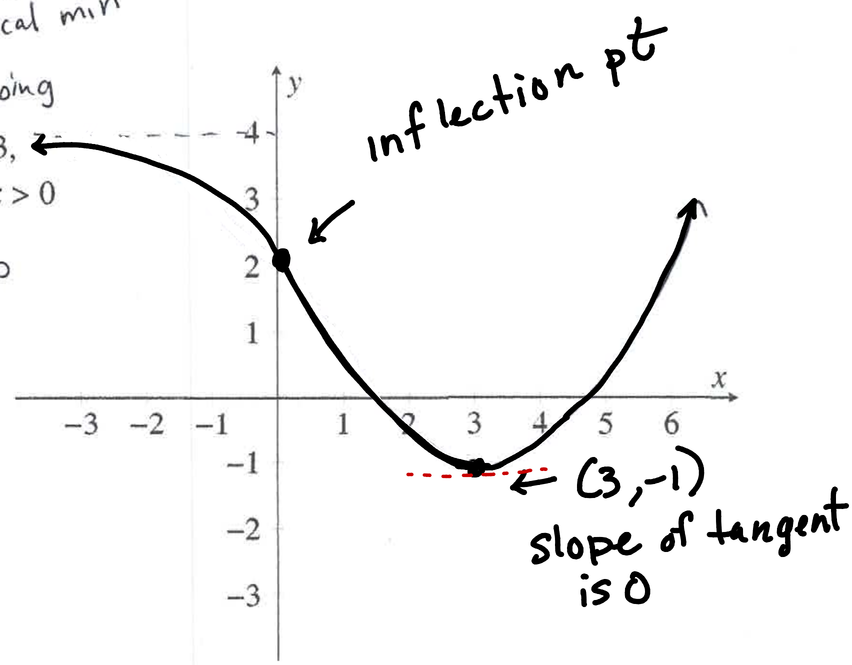
$f(3) = -1$, $f'(3) = 0$ increasing

decreasing $f'(x) < 0$ when $x < 3$, $f'(x) > 0$ when $x > 3$,

$f''(x) < 0$ when $x < 0$, $f''(x) > 0$ when $x > 0$

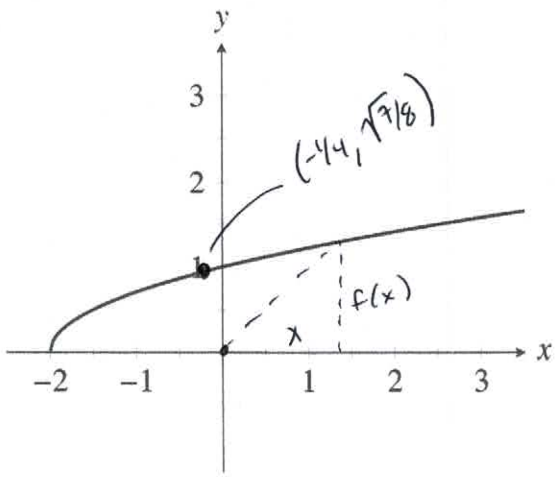
$\lim_{x \to -\infty} f(x) = 4$

cc down horizontal asymptote concave up



10. (12 points)

The graph of the function $f(x) = \sqrt{\frac{x}{2} + 1}$ is shown.



a. Let $G(x)$ be the square of the distance from the origin to a point on the graph of $y = f(x)$. Write an expression for $G(x)$.

$$x^2 + f(x)^2 = G(x)$$

$$G(x) = x^2 + \left(\sqrt{\tfrac{x}{2}+1}\right)^2$$

$$G(x) = x^2 + \frac{x}{2} + 1$$

b. Use the expression for $G(x)$ to find the closest point on the graph $y = f(x)$ to the origin.

$G'(x) = 2x + \frac{1}{2}$ $\quad$ $G'(x)=0$ when $x = -1/4$

sign of $G'(x)$: $-$ | $+$ at $-1/4$

$G(x)$ has a min at $x = -1/4$

$$f(-1/4) = \sqrt{\tfrac{-1/4}{2}+1} = \sqrt{7/8}$$

Closest point is $(-1/4, \sqrt{7/8})$

c. Show your result by adding a point, with coordinates, to the graph.