Math F251 Midterm 2 Spring 2021

Name: Solutions

Section: ☐ FXA (Sus)
☐ FXB (Maxwell)
☐ UX1 (Jurkowski)

Rules:

You have 60 minutes to complete the exam.

Partial credit will be awarded, but you must show your work.

A scientific or graphing calculator (without symbolic manipulation) is allowed.

A one page sheet of paper (8 1/2 in. x 11 in.) with handwritten notes on one side is allowed.

No other aids are permitted.

Place a box around your FINAL ANSWER to each question where appropriate.

Turn off anything that might go beep during the exam.

Academic Integrity Statement:

All students must affirm the following statements by initialing in the blanks provided. Students using their own paper must write out the statements in full.

____________ I will not seek or accept help from anyone.

____________ I will not use books, the internet or other disallowed aids.

____________ I understand correct answers without sufficient supporting work will be marked incorrect.

Problem	Possible	Score
1	10	
2	10	
3	10	
4	12	
5	10	
6	10	
Extra Credit	3	
Total	62	

1. (10 points)

Consider the function

$$f(t) = \frac{t}{1+t^2}.$$

a. Find the critical number(s) of $f(t)$ on the interval $[0,4]$

$$f'(t) = \frac{1\cdot(1+t^2) - t\cdot 2t}{(1+t^2)^2} = \frac{1+t^2-2t^2}{(1+t^2)^2} = \frac{1-t^2}{(1+t^2)^2} =$$

$$= \frac{(1-t)(1+t)}{(1+t^2)^2}$$

$f'(t)$ is defind for all x in $[0,4]$.

$$f'(t) = \frac{(1-t)(1+t)}{(1+t^2)^2} = 0 \Rightarrow \boxed{t=1,\ t=-1} \text{ are CP}$$

b. Find the absolute maximum and absolute minimum values of $f(t)$ on the interval $[0,4]$.

Let use the closed interval method.

1. $t=1$: $f(1) = \frac{1}{2}$

   $t=-1$: $f(-1) = -\frac{1}{2}$

2. $t=0$: $f(0)=0$

   $t=4$: $f(4) = \frac{4}{17}$

abs. max value: $f(1) = \frac{1}{2}$

abs. min value: $f(-1) = -\frac{1}{2}$

2. (10 points)

a. Find the linearization of $g(x) = \sqrt{x}$ at $a = 100$.

$$L(x) = g'(a)(x-a) + f(a)$$

$g(100) = 10$

$g'(x) = \frac{1}{2\sqrt{x}}$

$g'(100) = \frac{1}{20}$

$$L(x) = \frac{1}{20}(x-100) + 10$$

b. Use your result in part **a.** to approximate $\sqrt{100.5}$.

$$\sqrt{100.5} \approx L(100.5) = \frac{1}{20}(100.5-100) + 10 =$$

$$= \frac{1}{20}\cdot\frac{1}{2} + 10 = \frac{1}{40} + 10 = \frac{401}{40}$$

3. (10 points)

Evaluate the following limits. [Note: You should be careful to apply L'Hospital's rule **only** when appropriate.]

a. $\lim_{x\to 0} \frac{x^2}{1-\cos(x)}$ $=$ "$\frac{0}{0}$"

$$\lim_{x\to 0} \frac{x^2}{1-\cos(x)} \overset{L'H}{=} \lim_{x\to 0} \frac{2x}{0+\sin(x)} \overset{\frac{0}{0}}{\underset{L'H}{=}}$$

$$= \lim_{x\to 0} \frac{2}{\cos(x)} = 2$$

b. $\lim_{x\to 0^+} x\ln(x)$ $=$ "$0\cdot\infty$"

$$\lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0$$

c. $\lim_{\theta\to\pi} \frac{1-\cos(2\theta)}{\cos(\theta)}$ $= \frac{1-\cos(2\pi)}{\cos(\pi)} = \frac{0}{-1} = 0.$

4. (10 points)

Consider a secret function $f(x)$. We have computed for you

$$f'(x) = \frac{2(x^2-1)}{(x^2+1)^2} \quad \text{and} \quad f''(x) = \frac{4x(3-x^2)}{(x^2+1)}.$$

For full credit, show your work.

a. Find the intervals where $f(x)$ is increasing and decreasing.

$f(x)$ is increasing on I if $f'(x)>0$ on I
$f(x)$ is decreasing on I if $f'(x)<0$ on I

$f'(x) = \frac{2(x^2-1)}{(x^2+1)}$ f' is defined for all x in $\mathbb{R}$

We see that $f'(x)=0$ at $\boxed{x=\pm 1.}$ CP

$$\begin{array}{ccccccc} + & & - & & + & & f'(x) \\ \nearrow & -1 & \searrow & 1 & \nearrow & & f(x) \end{array}$$

Therefore,
f is $\nearrow$ on $(-\infty,-1)\cup(1,\infty)$
f is $\searrow$ on $(-1,1)$

b. Find the intervals where $f(x)$ is concave up and concave down.

$f''(x) = \frac{4x(3-x^2)}{(x^2+1)}$ f'' is defined for all x in $\mathbb{R}$

We see that $f''(x)=0$ at $x=0$ and $x=\pm\sqrt{3}$

$$\begin{array}{cccccccc} + & & - & & + & & - & f''(x) \\ \cup & -\sqrt{3} & \cap & 0 & \cup & \sqrt{3} & \cap & f(x) \end{array}$$

Therefore,
- f is concave up on $(-\infty,-\sqrt{3})\cup(0,\sqrt{3})$
- f is concave down on $(-\sqrt{3},0)\cup(\sqrt{3},\infty)$

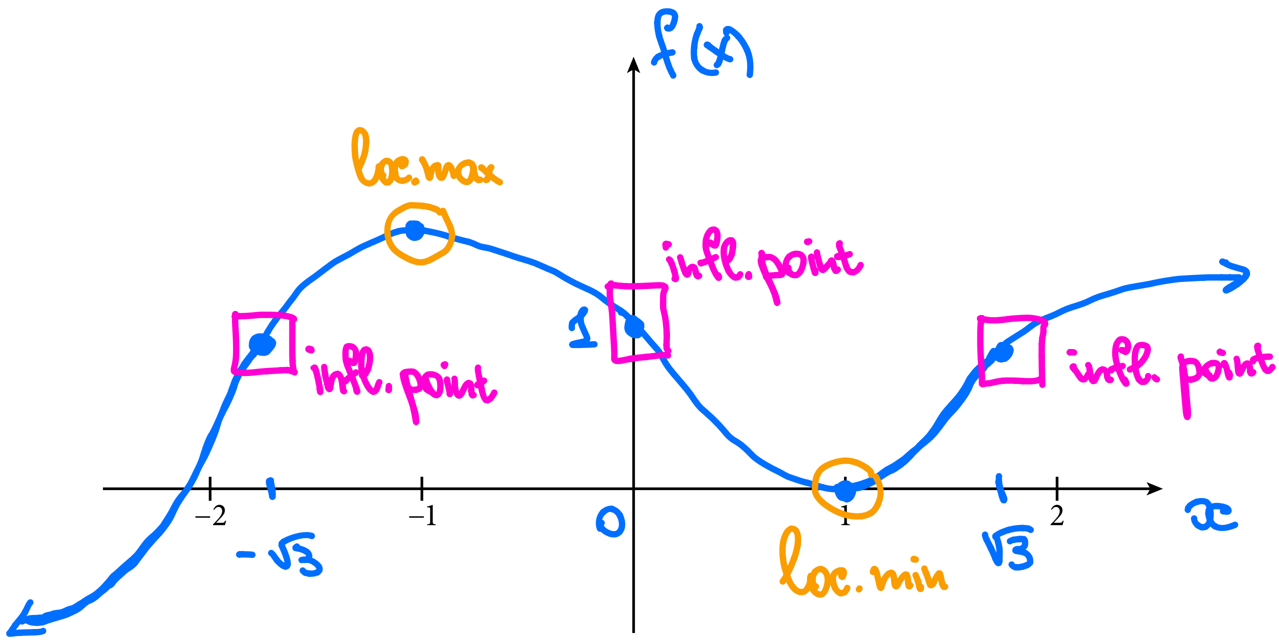
c. Classify, with justification, all critical points of $f(x)$.

By the 1st Derivative Test:
- at $x=-1$ $f(x)$ attains loc max value
- at $x=1$ $f(x)$ attains loc min value

- $x=-\sqrt{3},0,\sqrt{3}$ are inflection points ($f''(-\sqrt{3}) = f''(0) = f''(\sqrt{3})=0$ and f changes its concavity at $x=-\sqrt{3},0,\sqrt{3}$).

Continued....

d. In fact, for this secret function $f(0) = 1$ and $f(1) = 0$. Sketch the graph of any function that is consistent with this fact and the data from parts **a**-**c**. Denote any **points of inflection** on your graph with a square box.



5. (10 points)

An open faced metal box with a square bottom is to be constructed satisfying with the following constraints:

1. The volume is 1200 cm^2.
2. The bottom is made of silver costing 6 dollars per square centimeter, and the sides are made of copper costing 2 dollars per square centimeter.

a. If the base of the box is 10cm in width and the height of the box is 12cm, how much do the materials cost to build the box?

$P_B = \$6$

$P_S = \$2$

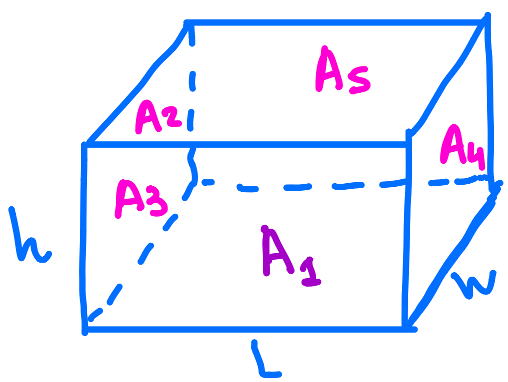
$C = P_B \cdot A_1 + P_S \cdot (A_2 + A_3 + A_4 + A_5)$

$C = 6 \cdot w^2 + 2 \cdot (h \cdot w + h \cdot w + hw + hw)$

$C = 6w^2 + 8hw = 6 \cdot 100 + 8 \cdot 120 = 1560 \$$

b. What dimensions of the box minimize the material costs? Include units in your answer.

$w, L, h > 0$



$w = 10$ cm

$h = 12$ cm

$w = L$

$V = 1200 \text{ cm}^3$

$V = h \cdot w^2 = 1200$

$h = \dfrac{1200}{w^2}$

$C = 6w^2 + 8hw$

$C = 6w^2 + 8 \cdot \dfrac{1200}{w^2} \cdot w$

$C(w) = C = 6w^2 + \dfrac{9600}{w} \to \min$

• $C'(w) = 12w - \dfrac{9600}{w^2} = 0 \Rightarrow \dfrac{12w^3 - 9600}{w^2} = 0 \Rightarrow \boxed{w \approx 9.3}$ (cm)

At $w = 0$ $C'(w)$ DNE, but $w > 0$.

<u>Justification:</u>

• $C''(w) = 12 + \dfrac{9600}{w^3} > 0$ at $w = 9.3$. Therefore, at $w = 9.3$ $C(w)$ attains its abs. min value.

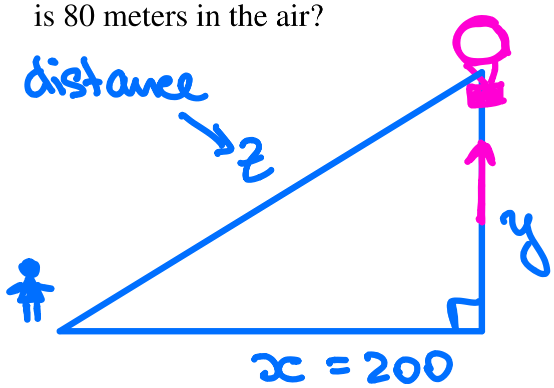
From this it follows that $L = w = 9.3$ (cm)

and $h = \dfrac{1200}{9.3^2} \approx 13.87$ (cm).

Answer: $w = L = 9.3$ (cm), $h = 13.87$ (cm).

6. (10 points)

A hot air balloon is rising straight up at a rate of 40 meters per minute. You are 200 meters away from the balloon's launch site. How fast is the distance between you and the balloon increasing when the balloon is 80 meters in the air?



$y = y(t)$

$z = z(t)$

Given:

$x = 200 (m)$

$\frac{dy}{dt} = 40 (m/min)$

Want to find:

$\frac{dz}{dt}$ - ? when $y = 80 (m)$

By Pythagorean Theorem:

$x^2 + y^2 = z^2$

Then,

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(z^2)$$

$$\cancel{2}y\frac{dy}{dt} = \cancel{2}z\frac{dz}{dt}$$

$$\frac{dz}{dt} = \frac{y}{z}\frac{dy}{dt}$$

$y = 80$, $x = 200$

$z^2 = 200^2 + 80^2$

$z \approx 215$

$$\boxed{\frac{dz}{dt} = \frac{80}{215} \cdot 40 \approx 15 (m/min)}$$

Extra Credit. (3 points)

Compute $\lim_{x\to\infty} x^{1/x}$. $=$ "∞^0"

$f(x) = x^{1/x}$

$\ln f(x) = \frac{1}{x}\ln x$

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{1}{x}\ln x \overset{0\cdot\infty}{=} \lim_{x\to\infty} \frac{\ln x}{x} =$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

$$\ln\left(\lim_{x\to\infty} f(x)\right) = 0 \Rightarrow \lim_{x\to\infty} f(x) = e^0 = \boxed{1}$$