Spring 2024 **Math F251X**

Calculus 1: Midterm 1

Name: Key

Section:
- ☐ 9:15am (James Gossell)
- ☐ 11:45am (Mohamed Nouh)
- ☐ async (Leah Berman)

Rules:

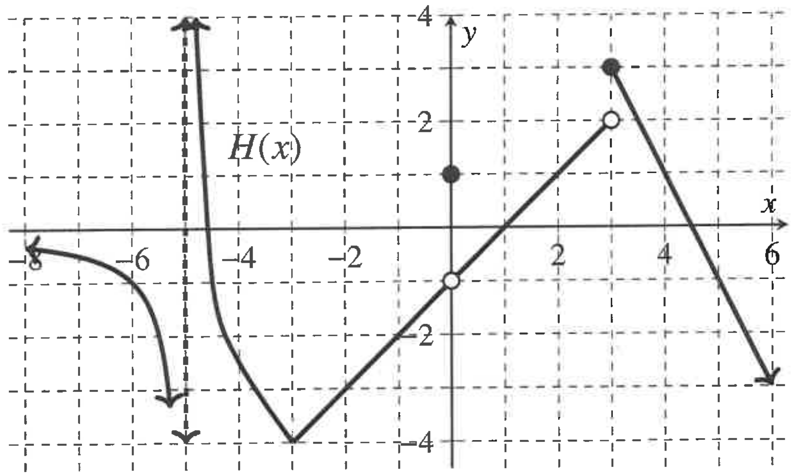
- Partial credit will be awarded, but you must show your work.
- You may have a single handwritten $3'' \times 5''$ notecard, both sides.
- Calculators are **not** allowed.
- Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question where appropriate.
- Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	16	
2	12	
3	10	
4	12	
5	12	
6	8	
7	10	
8	12	
9	8	
Extra Credit	5	
Total	100	

1. **(16 points)**

The entirety of a function $H(x)$ is shown below. Use the graph of $H(x)$ to answer each question below. If a limit is infinite, indicate that with ∞ or $-\infty$. If a value does not exist or is undefined, write **DNE**.



a. What is the **domain** of $H(x)$? Write your answer in **interval notation**.

domain = $(-\infty, -5) \cup (-5, \infty)$

b. $\lim_{x\to-5^-} H(x) =$ $-\infty$

c. $\lim_{x\to-5^+} H(x) =$ $+\infty$

d. $\lim_{x\to-5} H(x) =$ DNE

e. $\lim_{x\to3^-} H(x) =$ 2

f. $\lim_{x\to3^+} H(x) =$ 3

g. $H(3) =$ 3

h. $H'(4) =$ -2

i. $H(0) =$ 1

j. $\lim_{x\to0} H(x) =$ -1

k. $\lim_{x\to-3} H(x) =$ -4

l. $H(-3) =$ -4

m. Is $H'(-3)$ defined? Why or why not? Explain your answer in a few words.

No. It has a point. Basically $\lim_{x\to-3^-} \frac{H(x)-H(-3)}{x-(-3)}$ does not equal $\lim_{x\to-3^+} \frac{H(x)-H(-3)}{x-(-3)}$.

n. List the values of x in the interval $(-\infty, \infty)$ where $H(x)$ is NOT continuous. If $H(x)$ is continuous everywhere, write "none".

$x =$ $-5, 0, 3$

2. (12 points)

Compute the following limits. Show your work. Use limit notation where necessary; you will be graded both on your computation and on your correct use of notation.

If the limit does not exist, write **DNE** and a few words about why it does not exist. If the limit increases without bound, write ∞ or $-\infty$.

a. $\lim_{x\to 3} \frac{e^x+4}{x-8} = \frac{e^3+4}{3-8} = \boxed{\frac{e^3+4}{-5}}$

b. $\lim_{t\to 2^-} \frac{(t+2)(t+3)}{t(t-2)} = \frac{(2+2)(2+3)}{2(2^- -2)} = \frac{4\cdot 5}{2\cdot 0^-} \to \boxed{-\infty}$

c. $\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})}{h}\cdot\frac{(\sqrt{x+h}+\sqrt{x})}{(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$

$= \lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$

d. A function $f(x)$ has been numerically evaluated as follows:

x	1.001	1.0001	1.00001	0.99999	0.9999	0.999
$f(x)$	1.50150	1.50015	1.50002	1.49999	1.49985	1.49850

Estimate $\lim_{x\to 1} f(x) =$ 1.5

3. (10 points)

Consider the function

$$f(x) = \frac{3}{x^2} + 7.$$

Find $f'(2)$ using the **limit definition of the derivative** and show your work using all appropriate notation. No credit will be awarded for using other methods. Begin by writing down the limit definition of the derivative. You must write limits where necessary to receive full credit.

$$f'(2) = \lim_{h \to 0} \frac{\left(\frac{3}{(2+h)^2} + 7\right) - \left(\frac{3}{2^2} + 7\right)}{h} = \lim_{h \to 0} \frac{\frac{3}{(2+h)^2} - \frac{3}{4}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{12 - 3(2+h)^2}{4(2+h)^2}}{h} = \lim_{h \to 0} \frac{12 - 12 - 12h - 3h^2}{4h(2+h)^2} = \lim_{h \to 0} \frac{-3h(4+h)}{4h(2+h)^2}$$

$$= \lim_{h \to 0} \frac{-3(4+h)}{4(2+h)^2} = \frac{-3 \cdot 4}{4 \cdot 2^2} = \boxed{-\frac{3}{4}}$$

4. (12 points)

A certain population of rabbits can be modeled by a function $P(t)$, where t measures time, in years, since 2018.

a. Suppose you are told that $P(5) = 285$. Write a sentence explaining what this means in the context of the problem, and use units in your answer.

In 2023, there are 285 rabbits.

b. Suppose, in addition, you are told that $P(0) = 20$. What is the average rate of population change between 2018 and 2023? Do a computation to determine this quantity, and write your answer in a sentence, including units.

In 2018, there were 20 rabbits. Between 2018 and 2023, the average rate of population change was: $\frac{285-20}{5} = \frac{265}{5} = 53$ rabbits per year

c. The quantity $P'(5) = 85$. Write a sentence explaining what this means in the context of the problem, and include units in your answer.

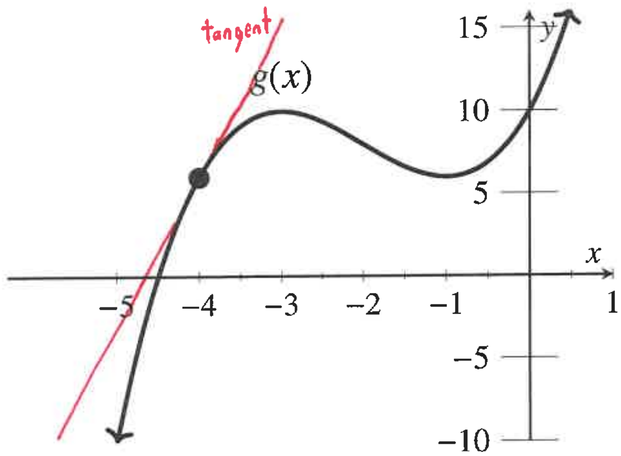
In 2023, the rabbit population was increasing at a rate of 85 rabbits per year

d. Suppose that $P'(t) < 0$ for $t \geq 8$. Write a sentence explaining what this means in the context of the problem.

After 2026, the rabbit population will be declining.

5. (12 points)

Consider the function $g(x) = x^3 + 6x^2 + 9x + 10$ whose graph is shown below.



a. Draw and **label** (with the word "tangent") the tangent line to the graph at the point $(-4, g(-4)) = (-4, 6)$ shown on the graph.

b. Find $g'(x)$. (You do not need to use the limit definition of the derivative to answer this problem.)

$g'(x) = 3x^2 + 12x + 9$

c. Write down an equation for the tangent line at the point $(-4, 6)$. Your equation should use point-slope form: that is, it should be of the form $y = m(x - x_1) + y_1$.

$m = g'(-4) = 3(-4)^2 + 12(-4) + 9 = 9$

equation of tangent line: $y = 9(x+4) + 6$

d. Do a computation to determine (exactly) the x-coordinates of all points on the graph where there is a horizontal tangent line. Show your work.

Set $g'(x)$ equal to 0: $3x^2 + 12x + 9 = 0 \Rightarrow x^2 + 4x + 3 = 0 \Rightarrow (x+3)(x+1) = 0$

$\Rightarrow x = -3$ or -1

6. (8 points)

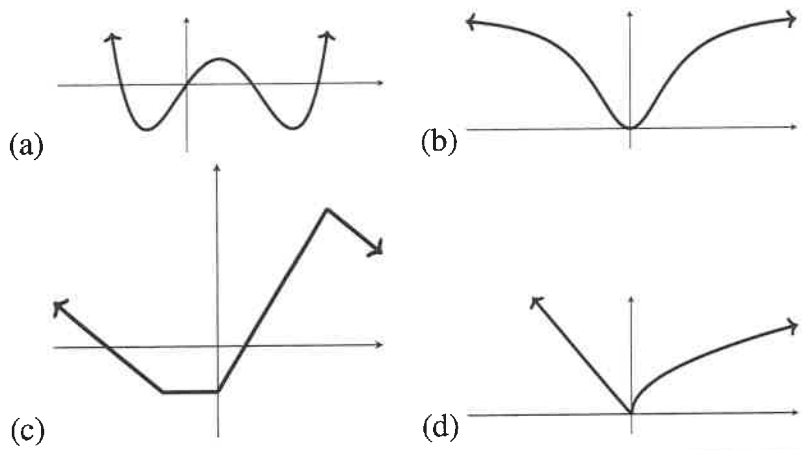
Match the graph of each function (a) – (d) with the graph of its derivative, chosen from the list of graphs I – IX. Write your answer in the blanks below.

Your answers:

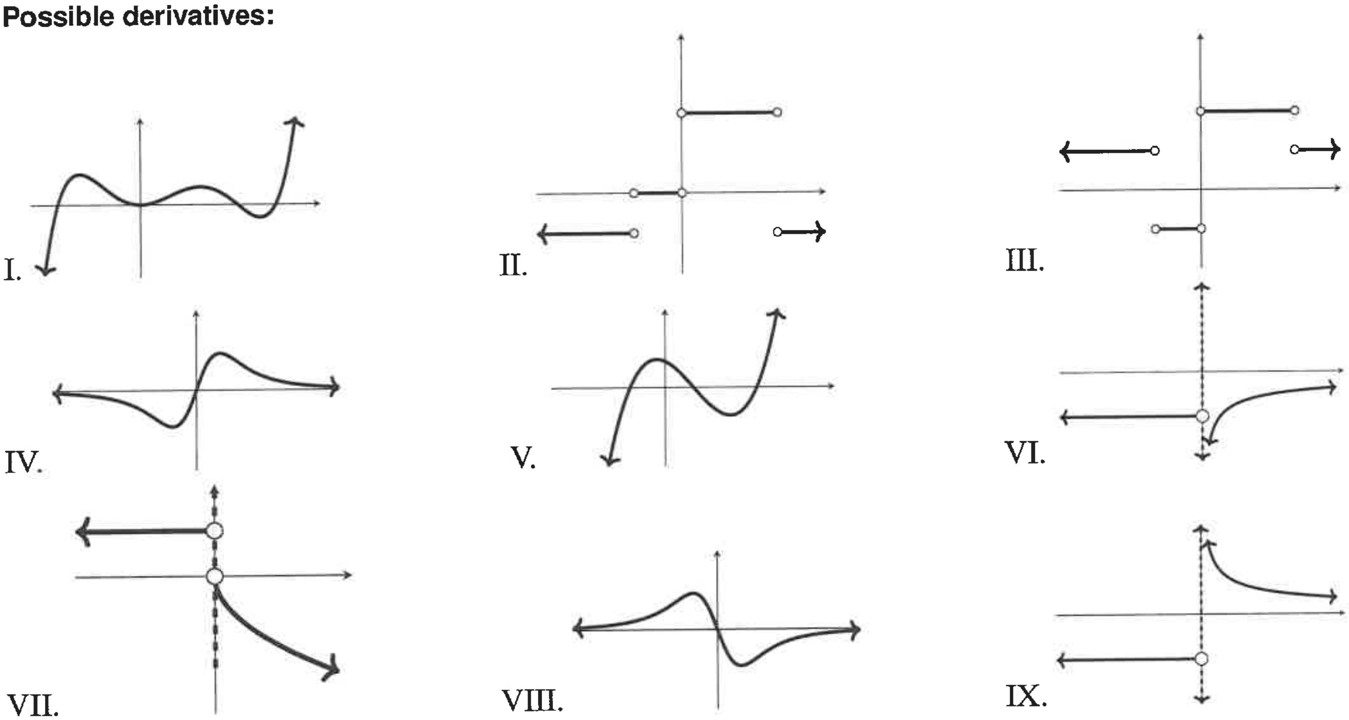
i. The derivative of graph (a) is <u>V</u>. **ii.** The derivative of graph (b) is <u>IV</u>.

iii. The derivative of graph (c) is <u>II</u>. **iv.** The derivative of graph (d) is <u>IX</u>.

The graphs:



Possible derivatives:



7. (10 points)

Let

$$f(x) = \begin{cases} \dfrac{4^x}{x+x^2} & x < 1 \\ 1 & x = 1 \\ \dfrac{2-2x}{x-x^2} & x > 1 \end{cases}$$

a. Evaluate $\lim_{x \to 1^-} f(x)$. Show supporting work, including correct use of limit notation.

$$\lim_{x \to 1^-} f(x) = \frac{4^1}{1+1^2} = \frac{4}{2} = \boxed{2}$$

b. Evaluate $\lim_{x \to 1^+} f(x)$. Show supporting work, including correct use of limit notation.

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{2-2x}{x-x^2} = \lim_{x \to 1^+} \frac{2(1-x)}{x(1-x)} = \lim_{x \to 1^+} \frac{2}{x} = \frac{2}{1} = \boxed{2}$$

c. Evaluate $f(1)$. $f(1) = \boxed{1}$

d. Based on your answers to parts (a), (b) and (c), **check the true statement(s) below:**

- [ ] f is continuous at $x = \cancel{0}$. 1
- [x] f has a removable discontinuity at $x = \cancel{0}$. 1
- [ ] f has a jump discontinuity at $x = \cancel{0}$. 1
- [ ] f has an infinite discontinuity at $x = \cancel{0}$. 1
- [ ] None of the above.

8. (12 points)

For each of the following functions, compute the derivative. **You do not need to simplify your answers.** Your answer must begin with $f'(x)$, $\frac{df}{dx}$, $\frac{dy}{dx}$, y', or similar notation, as appropriate to the problem.

a. $f(x) = x^6 - 7x^4 + \dfrac{1}{x^2} + \cos(2)$

$f'(x) = 6x^5 - 28x^3 - 2x^{-3}$

b. $g(\theta) = (2\theta + \pi)\sin(\theta)$

$g'(\theta) = 2\sin\theta + (2\theta + \pi)\cos\theta$

c. $h(t) = \dfrac{t^{\frac{5}{2}} + t - 4}{\sqrt{t}}$ $= t^2 + \sqrt{t} - 4t^{-1/2}$

$h'(t) = 2t + \frac{1}{2}t^{-\frac{1}{2}} + 2t^{-3/2}$

d. $k(x) = \dfrac{3x^2}{x^2 + 2}$

$k'(x) = \dfrac{6x(x^2+2) - 3x^2(2x)}{(x^2+2)^2}$

9. (8 points)

A particle moves back and forth along a coordinate axis in such a way that its **position** at time t is given by the function

$$s(t) = t - \cos(t).$$

The position of the particle is measured in millimeters, and time is measured in seconds.

a. Determine the velocity function $v(t)$ and the acceleration function $a(t)$ of the particle.

$v(t) = s'(t) = 1 + \sin t$

$a(t) = s''(t) = \cos t$

b. What is the initial velocity of the particle? Give units in your answer.

$v(0) = 1 + \sin(0) = 1$ mm/sec

c. At time $t = \frac{\pi}{6}$ is the particle speeding up or slowing down? Explain how you know with a computation and some words.

$v(\frac{\pi}{6}) = 1 + \sin(\frac{\pi}{6}) = 1 + \frac{1}{2} = \frac{3}{2}$

$a(\frac{\pi}{6}) = \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

Since both $v(\frac{\pi}{6})$ and $a(\frac{\pi}{6})$ are positive, the particle is <u>speeding up</u> at $t = \frac{\pi}{6}$ seconds.

d. On the time interval $[0, \pi]$, does the particle ever stop? Explain your answer with a computation and some words.

It never stops. This is because $v(t) = 1 + \sin t$ is never 0 on $[0, \pi]$.

Extra Credit: (5 points) Consider the function $f(x) = \dfrac{2^x}{1-x}$.

i. What is $f(0)$?

$$f(0) = \frac{2^0}{1-0} = 1$$

ii. What is $f(3)$?

$$f(3) = \frac{2^3}{1-3} = -4$$

iii. Can we use the Intermediate Value Theorem to conclude that $f(x) = 0$ for some x in the interval $[0, 3]$? If so, write an explanation for how the Intermediate Value Theorem lets us conclude this. If not, explain why not.

We cannot conclude that $f(x) = 0$ on $[0,3]$. This is because $f(x)$ is not continuous at $x = 1$, and 1 is in the interval $(0,3)$.