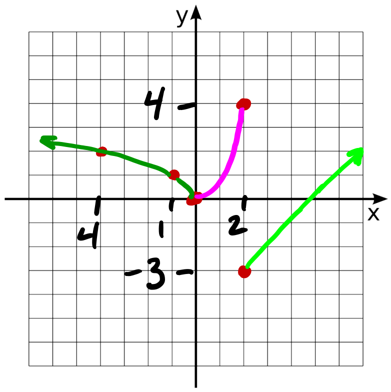
Final Review - Chapter 2
(Limits, + Continuity + L'Hospital's Rule)

Example 1: Sketch the graph of $f(x) = \begin{cases} \sqrt{-x}, & \text{if } x < 0 \\ x^2 & \text{if } 0 < x \leq 2 \\ x - 5, & \text{if } x > 2 \end{cases}$ and give the interval on which f is continuous. At what numbers is f continuous from the right, left or neither?



a) $\lim_{x\to 0^-} f(x) = 0$

b) $\lim_{x\to 0^+} f(x) = 0$

c) $\lim_{x\to 0} f(x) = 0$

d) $\lim_{x\to 2^-} f(x) = 4$

e) $\lim_{x\to 2^+} f(x) = -3$

f) $\lim_{x\to 2} f(x)$ DNE

- Find limits using factoring, algebra, conjugates.

Example 2: Find the following limits:

a) $\lim_{x\to -1^-} f(x)$ for $f(x) = \begin{cases} x^2 - 1 & \text{for } x < 1 \\ 2x + 3 & \text{for } x \geq 1 \end{cases}$

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} x^2 - 1 = (-1)^2 - 1 = 1 - 1 = 0$

b) $\lim_{x\to 0^+} f(x)$ where $f(x) = \begin{cases} x^2 + 4 & \text{for } x > 0 \\ 2\cos(x) + 5 & \text{for } x \leq 0 \end{cases}$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 + 4 = 4$

Example 3: Find the following limits:

a) $\lim_{x\to 1} e^{x-1} \sin\left(\frac{\pi x}{2}\right)$

$= e^0 \cdot \sin\left(\frac{\pi}{2}\right)$

$= 1 \cdot 1 = 1$

b) $\lim_{x\to 0} \frac{5x^2}{1 - \cos x} \overset{H}{=} \lim_{x\to 0} \frac{10x}{\sin x} \overset{H}{=} \lim_{x\to 0} \frac{10}{\cos x} = 10$

form $\frac{0}{0}$

form $\frac{0}{0}$

$2(x^2-9)$

$\begin{array}{r} 16 \\ \times 3 \\ \hline 48 \end{array}$

Example 4: Find the following limits:

a) $\displaystyle \lim_{x\to 3} \frac{2x^2-18}{x^2+x-12}$

$$= \lim_{x\to 3} \frac{2(x-3)(x+3)}{(x-3)(x+4)}$$

$$= \lim_{x\to 3} \frac{2(x+3)}{x+4} = \frac{2\cdot 6}{7}$$

$$= \frac{12}{7}$$

b) $\displaystyle \lim_{h\to 0} \frac{(4+h)^3-64}{h} = \lim_{h\to 0} \frac{64+48h+12h^2+h^3-64}{h}$

$$= \lim_{h\to 0} \frac{48h+12h^2+h^3}{h} = \lim_{h\to 0} 48+12h+h^2 = 48$$

Example 5: Find the following limits:

a) $\displaystyle \lim_{x\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x} = \lim_{x\to -4} \frac{\frac{x+4}{4x}}{4+x}$

$$= \lim_{x\to -4} \frac{1}{4x} = \frac{-1}{16}$$

b) $\displaystyle \lim_{x\to -4} \frac{\sqrt{x^2+9}-5}{x^2+2x-8} \cdot \frac{\sqrt{x^2+9}+5}{\sqrt{x^2+9}+5}$

$$= \lim_{x\to -4} \frac{x^2+9-25}{(x+4)(x-2)(\sqrt{x^2+9}+5)}$$

$$= \lim_{x\to -4} \frac{\cancel{(x+4)}(x-4)}{\cancel{(x+4)}(x-2)(\sqrt{x^2+9}+5)} = \frac{-8}{-6(10)} = \frac{2}{15}$$

- Find infinite limits. As in the limit is equal to plus or minus infinity or has an infinite discontinuity.
- Find limits at infinity. This means x goes to plus or minus infinity.

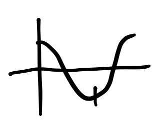
Example 6: Find the following limits:

a) $\displaystyle \lim_{x\to 5^-} \frac{e^x}{(x-5)^3} = -\infty$

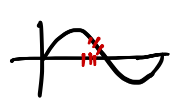
as $x\to 5^-$, $x-5\to 0^-$

and $e^x \to e^5 > 0$

b) $\displaystyle \lim_{x\to \pi^-} \cot x = \lim_{x\to\pi^-} \frac{\cos x}{\sin x} = -\infty$



$\cos x \to -1$



$\sin x \to 0^+$

Example 7: Find the following limits.

a) $\displaystyle\lim_{x\to\infty} \frac{4x^4+5}{(x^2-2)(2x^2-1)}$

$$= \lim_{x\to\infty} \frac{4x^4+5}{2x^4-5x^2+2} \cdot \frac{1/x^4}{1/x^4}$$

$$= \lim_{x\to\infty} \frac{4+5/x^4}{2-5/x^2+2/x^4}$$

$$= \frac{4+0}{2-0+0} = 2$$

b) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{9x^6-x}}{x^3+1} = \lim_{x\to+\infty} \frac{\sqrt{9(-x)^6-(-x)}}{(-x)^3+1}$

replace x w/ -x

$$= \lim_{x\to+\infty} \frac{\sqrt{9x^6+x}}{-x^3+1} \cdot \frac{1/x^3}{1/x^3}$$

Don't forget that $x^3 = \sqrt{x^6}$ if $x \geq 0$

$$= \lim_{x\to\infty} \frac{\sqrt{9+1/x^5}}{-1+1/x^3} = \frac{\sqrt{9+0}}{-1+0} = -3$$

Example 8: Find the following limits.

a) $\displaystyle\lim_{x\to\infty} \frac{x+x^3+x^5}{1-x^2+x^4} \cdot \frac{1/x^4}{1/x^4}$

$$= \lim_{x\to\infty} \frac{1/x^3+1/x+x}{1/x^4-1/x^2+1} = \infty$$

b/c $x\to\infty$, and the denominator approaches 1.

b) $\displaystyle\lim_{x\to\infty} \frac{x^3-2x+3}{5-2x^4} \cdot \frac{1/x^4}{1/x^4} = \lim_{x\to\infty} \frac{\frac{1}{x}-\frac{2}{x^3}+\frac{3}{x^4}}{\frac{5}{x^4}-2}$

$$= \frac{0-0+0}{0-2} = \frac{0}{-2} = 0$$

Example 9: Find the following limits.

a) $\displaystyle\lim_{x\to\infty} \sec\left(\frac{x^2}{x^3-2}\right)$

$$= \sec\left(\lim_{x\to\infty} \frac{x^2}{x^3-2}\right)$$

$$= \sec(0) = 1$$

b) $\displaystyle\lim_{x\to0^+} \arctan(1/x) = \pi/2$

Reason:

As $x\to0^+$, $\frac{1}{x}$ approaches positive infinity. As $z\to\infty$, $\arctan z$ approach $\pi/2$.

Example 10: Find the following limits using l'Hospital's rule. I won't tell you explicitly to do this on the exam. You will have to know when you can/ cannot apply this rule.

a) $\lim_{x \to \infty} \frac{1-e^x}{1+2e^x} \overset{H}{=} \lim_{x \to \infty} \frac{-e^x}{2e^x}$

form $\frac{\infty}{\infty}$

$= \lim_{x \to \infty} \frac{-1}{2} = -\frac{1}{2}$

b) $\lim_{h \to 0} \frac{\sin h}{h \cos h} \overset{H}{=} \lim_{h \to 0} \frac{\cosh}{1 \cdot \cosh - h \sin x}$

form $\frac{0}{0}$

$= \frac{1}{1-0} = 1$

- Know and apply the defintion of continuity.
- Determine where a function is discontinuous and why.
- Determine the value of a constant that makes a function continuous.

Definition of Continuity A function f is continuous at c if the following three conditions are met:

1. $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$ or $\lim_{x \to c} f(x)$ exists
2. $f(c)$ exists
3. $\lim_{x \to c} f(x) = f(c)$

Example 11: Find all points of discontinuity of $h(x) = \dfrac{x-4}{x^2-x-12}$ and explain why the points are discontinuous and state if they are removable or non-removable.

$x^2-x-12=(x-4)(x+3)$

So $h(x)$ fails to exist at $x=-3$ and $x=4$.

$\lim_{x \to 4} h(x) = \lim_{x \to 4} \frac{x-4}{(x-4)(x+3)} = \lim_{x \to 4} \frac{1}{x+3} = \frac{1}{7}$ ✓

$\lim_{x \to -3^+} \frac{1}{x+3} = +\infty$; $\lim_{x \to -3^-} \frac{1}{x-3} = -\infty$

Answer:

h is discontinuous at $x=-3$ and $x=4$.

not removable

removable.

Example 12: Find the numbers, if any, at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither?

$$f(x) = \begin{cases} x^2+1 & \text{if } x < 0 \\ e^x & \text{if } 0 \le x \le 2 \\ 6x-7 & \text{if } x > 2 \end{cases}$$

at $x=0$:

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2+1 = 1$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^x = e^0 = 1$

So $\lim_{x \to 0} f(x) = 1 = e^0 = f(0)$

So $f(x)$ is continuous at $x=0$.

at $x=2$:

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} e^x = e^2$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 6x-7 = 12-7 = 5$

So $\lim_{x \to 2} f(x)$ does not exist.

So $f(x)$ is not continuous at $x=2$.

Because $f(2) = e^2 = \lim_{x \to 2^-} f(x)$,

$f(x)$ is continuous from the left at $x=2$.

Example 13: Determine the value of b such that the function $f(x) = \begin{cases} x^2+bx & x \le 1 \\ 3\cos(\pi x) & x > 1 \end{cases}$ is continuous on the entire real line.

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2+bx = 1+b$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 3\cos(\pi x) = 3\cos(\pi) = -3$

For continuity at $x=1$, we need $1+b=-3$. $\boxed{\text{So } b=-4}$

Example 14: Determine the values of a and b that will make the function $f(x) = \begin{cases} x+1 & \text{if } 1 < x < 3 \\ x^2+ax+b & \text{if } |x-2| \ge 1 \end{cases}$ continuous on the entire real number line.

$|x-2| \ge 1$ means

$x-2 \ge 1$ or $x-2 \le -1$

$x \ge 3$ or $x \le 1$

at $x=1$, we need $1+1 = 1^2 + a \cdot 1 + b$

So $a+b=1$

at $x=3$, we need: $3+1 = 3^2+3a+b$.

So $3a+b=-5$

Now:

$$\begin{array}{r} 3a+b=-5 \\ -(\;a+b=\;1) \\ \hline 2a\quad=-6 \\ a\quad=-3 \end{array}$$

So $b=4$

Homework: Ch. 2 Rvw, Pg 165 # 1-19 odds, 29 and Pg 311 # 15 - 19 odds, 33 Due Friday by 10 AM