Final Review - Chapter 5
(Antiderivatives and Applications of Anti-Differentiation)

Example 1: Find the most general antiderivative of the function.

a) $g(x) = \frac{1}{x} + \frac{1}{x^2+1}$

$G(x) = \ln|x| + \arctan x + C$

b) $f(x) = \frac{x^2+\sqrt{x}}{x} = x + x^{-1/2}$

$F(x) = \frac{1}{2}x^2 + 2x^{1/2} + C$

Example 2: Given $f''(x) = 5x^3 + 6x^2 + 2$, $f(0) = 3$, $f(1) = -2$, find $f(x)$.

$f'(x) = \int (5x^3+6x^2+2)\,dx = \frac{5}{4}x^4 + 2x^3 + 2x + C$

$f(x) = \int f'(x)\,dx = \int \left(\frac{5}{4}x^4 + 2x^3 + 2x + C\right)dx = \frac{1}{4}x^5 + \frac{1}{2}x^4 + x^2 + Cx + D$

$3 = f(0) = D$. So $D = 3$.

$-2 = f(1) = \frac{1}{4} + \frac{1}{2} + 1 + C + 3$

$C = -6\frac{3}{4} = -27/4$

ans:

$f(x) = \frac{1}{4}x^5 + \frac{1}{2}x^4 + x^2 - \frac{27}{4}x + 3$

Example 3: A particle is moving with $v(t) = 2t - 1/(1+t^2)$ and $s(0) = 1$. Fin the position of the particle.

$v(t) = 2t - \frac{1}{1+t^2}$

$s(t) = \int v(t)\,dt = \int \left(2t - \frac{1}{1+t^2}\right)dt = t^2 - \arctan t + C$

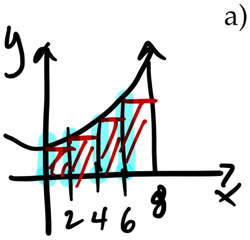
$1 = s(0) = 0^2 - \arctan 0 + C = C$. So $C = 1$

So answer: $s(t) = t^2 - \arctan t + 1$

Example 4: Estimate the area under the curve $y = x^2 + 2$ on the interval $[0, 8]$ using 4 sub-intervals and the method given below.

$$\begin{array}{r} 30 \\ +62 \\ \hline 92 \end{array}$$

a) left endpoints.



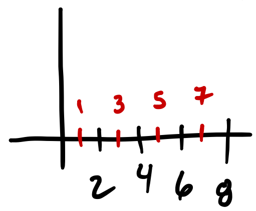
$$\begin{array}{r} 40 \\ +24 \\ \hline 64 \end{array}$$

$L_4 = 2\big(f(0)+f(2)+f(4)+f(6)\big)$

$= 2(2+6+18+38)$

$= 2(64) = 128$

b) midpoints.



$M_4 = 2\big(f(1)+f(3)+f(5)+f(7)\big)$

$= 2(3+11+27+51)$

$= 2(92) = 184$

Example 5: Evaluate the following definite integrals.

a) $\displaystyle\int_0^{\pi/4} \frac{\sec^2 t}{\tan t + 1}\,dt$

$= \ln(\tan t + 1)\Big|_0^{\pi/4}$

$= \ln\left(\tan\frac{\pi}{4} + 1\right) - \ln(\tan 0 + 1)$

$= \ln 2 - \ln 1 = \ln 2$

b) $\displaystyle\int_1^4 \frac{x-2}{\sqrt{x}}\,dx = \int_1^4 \left(x^{1/2} - 2x^{-1/2}\right)dx$

$= \frac{2}{3}x^{3/2} - 4x^{1/2}\Big]_1^4$

$= \left(\frac{2}{3}(4)^{3/2} - 4(4)^{1/2}\right) - \left(\frac{2}{3} - 4\right)$

$= \frac{16}{3} - 8 - \frac{2}{3} + 4 = \frac{14}{3} - 4 = \frac{2}{3}$

$\uparrow$

$-\frac{12}{3}$

Example 6: Find the most general anti-derivaatives.

a) $\int \left(\sec x \tan x + \frac{2}{\sqrt{1-x^2}} \right) dx$

$= \sec x + 2 \arcsin x + C$

b) $\int \frac{x}{(x-2)^3} dx = \int (u+2)\, u^{-3} du = \int u^{-2} + 2u^{-3} du$

let $u = x-2$

$du = dx$

$x = u+2$

$= -u^{-1} - u^{-2} + C$

$= -(x-2)^{-1} - (x-2)^{-2} + C$

Example 7: Find the most general anti-derivatives.

a) $\int \frac{\sin(1/x)}{x^2} dx = -\int \sin u \, du$

let $u = \frac{1}{x} = x^{-1}$

$du = -x^{-2} dx$

$-du = x^{-2} dx = \frac{dx}{x^2}$

$= \cos u + C$

$= \cos(x^{-1}) + C$

b) $\int \frac{\cos^{-1} x}{\sqrt{1-x^2}} dx = \int u \, du = \frac{1}{2} u^2 + C$

let $u = \arccos x$

$du = \frac{1}{\sqrt{1-x^2}} dx$

$= \frac{1}{2} (\arccos x)^2$

Example 8: Find the derivative of the following functions.

a) $F(x) = \int_2^{x^3} \sqrt{1+t^4} dt$

$F'(x) = \sqrt{1 + (x^3)^4} \cdot 3x^2$

$= 3x^2 \sqrt{1 + x^{12}}$

b) $H(x) = \int_{e^x}^{x^2} \sec t dt = \int_{e^x}^{0} \sec t \, dt + \int_0^{x^2} \sec t \, dt$

$= -\int_0^{e^x} \sec t \, dt + \int_0^{x^2} \sec t \, dt$.

Now $H'(x) = -(\sec e^x) \cdot e^x + (\sec x^2) 2x$

$= -e^x \sec(e^x) + 2x \sec x^2$.

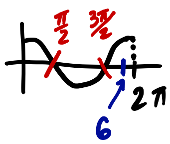
Example 9: A particle moves along a line with velocity function $v(t) = \cos t$, where v is measured in meters per second.

(a) Find the displacement over the time interval $[0, 6]$

$$\text{displacement} = \int_0^6 \cos t \, dt = \sin t \Big]_0^6 = \sin 6 - \sin 0 = \sin 6.$$

(b) Find the total distance traveled during the time interval $[0, 6]$

$$\text{Total distance} = \int_0^6 |\cos t| \, dt = \int_0^{\pi/2} \cos t \, dt - \int_{\pi/2}^{3\pi/2} \cos t \, dt + \int_{3\pi/2}^{6} \cos t \, dt$$



$$= \sin t \Big]_0^{\pi/2} - \left(\sin t \Big]_{\frac{\pi}{2}}^{3\pi/2}\right) + \left(\sin t \Big]_{\frac{3\pi}{2}}^{6}\right)$$

$$= \left[\sin\left(\tfrac{\pi}{2}\right) - \sin 0\right] - \left[\sin\left(\tfrac{3\pi}{2}\right) - \sin\left(\tfrac{\pi}{2}\right)\right] + \left[\sin 6 - \sin \tfrac{3\pi}{2}\right]$$

$$= 1 - [-1 - 1] + \sin 6 + 1$$

$$= 1 + 2 + \sin 6 + 1$$

$$= 4 + \sin 6$$

Example 10: A bacteria population is 4000 at time $t = 0$ and its rate of growth is 1000×2^t bacteria per hour after t hours. What is the population after one hour?

Let $P(t)$ be the population at time t.

Thus $P(0) = 4000$ and $P'(t) = 1000 \cdot 2^t$.

$$\text{So } P(1) = 4000 + \int_0^1 1000 \cdot 2^t \, dt = 4000 + 1000 \cdot \frac{1}{\ln 2} \cdot 2^t \Big]_0^1$$

$$= 4000 + \frac{1000}{\ln 2}\left(\underbrace{2^1 - 2^0}_{2-1=1}\right) = 4000 + \frac{1000}{\ln 2} \text{ bacteria}$$