1. Consider the function $f(x)$ and its derivatives:

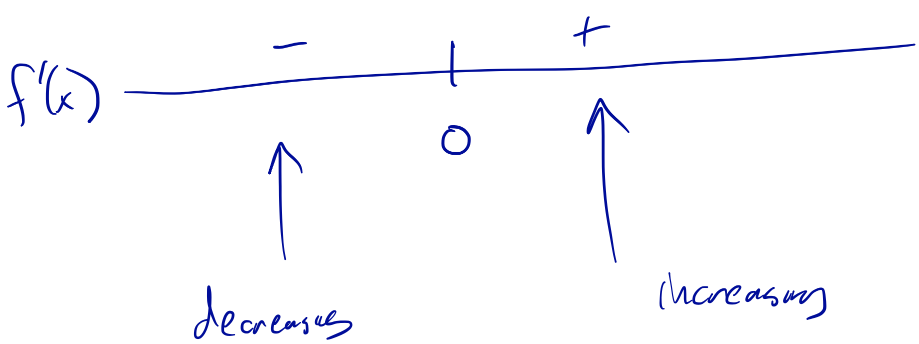
$$f(x) = \frac{e^x}{1+x}$$
$$f'(x) = \frac{xe^x}{(1+x)^2}$$
$$f''(x) = \frac{e^x(x^2+1)}{(1+x)^3}.$$

a. Find the critical numbers of $f(x)$.

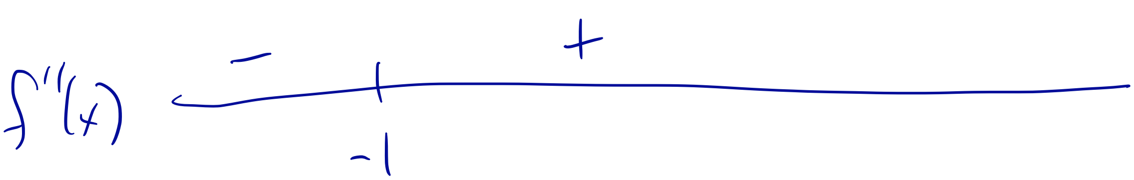
$f'(x) = 0$: $x = 0$ only

(function not defined at $x=-1$)

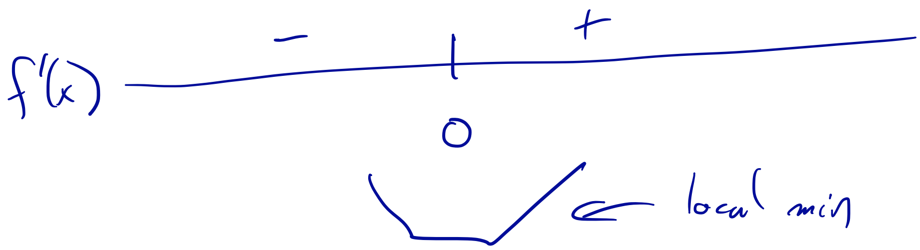
b. Find the open intervals on which the function is increasing or decreasing.



c. Find the open intervals on which the function is concave up or concave down.



d. Classify all critical points – using the first derivative test.



e. Classify all critical points – using the second derivative test.

$f''(0) = 1 > 0$

local min

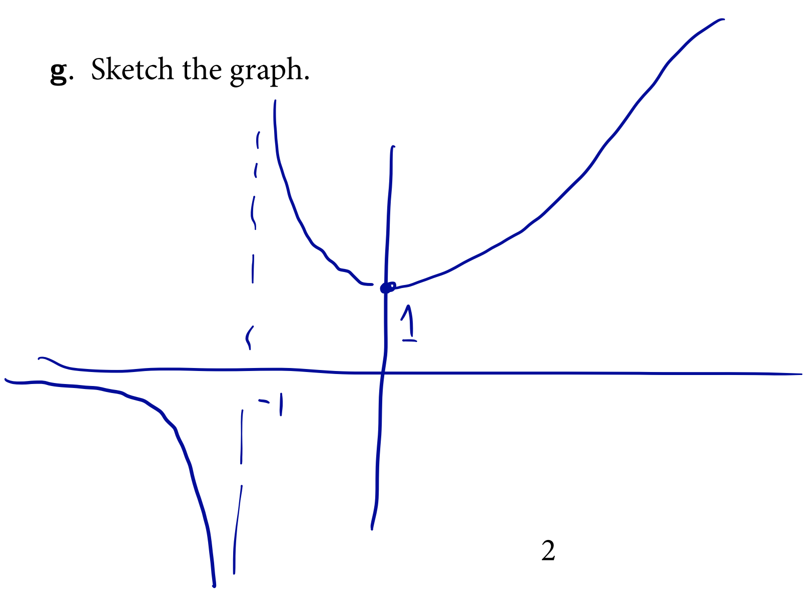
f. Find the inflection points.

None

$$\lim_{x \to \infty} \frac{e^x}{1+x} = \lim_{x \to \infty} \frac{e^x}{1} = \infty$$

$$\lim_{x \to -\infty} \frac{e^x}{1+x} = \frac{0}{-\infty} = 0^-.$$

g. Sketch the graph.



2. Find the linearization of $f(x) = \sqrt{x}$ at $a = 4$ and use it to estimate $\sqrt{4.1}$.

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$L(x) = f(a) + f'(a)(x-a)$$
$$= \sqrt{4} + \frac{1}{2\sqrt{4}}(x-4)$$
$$= 2 + \frac{1}{4}(x-4)$$

$$\sqrt{4.1} \approx L(4.1) = 2 + \frac{1}{40}$$

3. Show that the point $(2, 3)$ lies on the curve $x^2 + xy - y^2 = 1$. Then find the slope of the tangent line to the curve at that point.

$$2^2 + 2\cdot 3 - 3^2 = 4 + 6 - 9 = 1 \checkmark$$

$$2x + y + xy' - 2yy' = 0$$

$$y'(x - 2y) = -2x - y$$
$$y' = \frac{-2x-y}{x-2y} = \frac{-2\cdot 2 - 3}{2 - 2\cdot 3} = \frac{-7}{-4} = \frac{7}{4}$$

4. A ball of metal is being heated in an oven, and its radius is increasing at a rate of 0.1 cm/min. At what rate is the ball's volume increasing when its radius is 3 cm?

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi \cdot 9 \cdot \frac{1}{10} = \frac{36}{10}\pi \quad cm^3/min$$

5. Evaluate the following limits.

$\lim_{x \to 0} \frac{1 + x - e^x}{\sin x}$

$$\overset{0/0}{=} \lim_{x \to 0} \frac{1-e^x}{\cos(x)} = \frac{0}{1} = 0$$

$\lim_{x \to 0^+} (1 + 2x)^{1/x}$

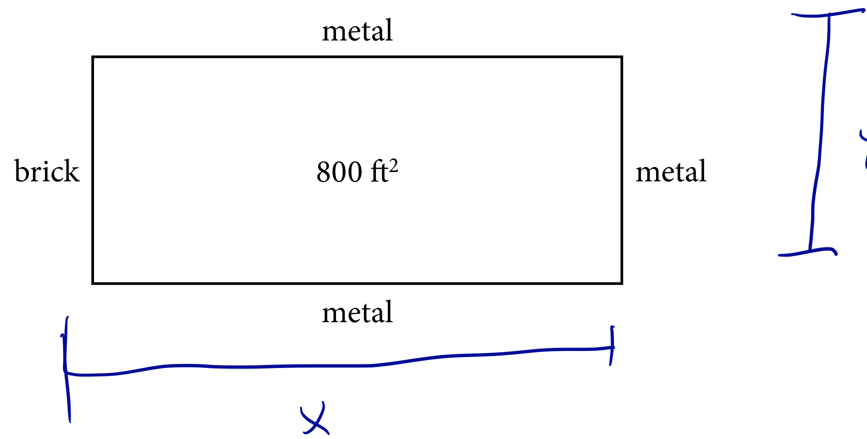
$$y = (1+2x)^{1/x}$$

$$\ln(y) = \frac{1}{x}\ln(1+2x)$$

$$\lim_{x \to 0^+} \frac{\ln(1+2x)}{x} \overset{0/0}{=} \lim_{x \to 0^+} \frac{\frac{1}{1+2x} \cdot 2}{1} = 2$$

$$\lim_{x \to 0^+} y = \lim_{x \to 0^+} e^{\ln(y)} = e^2$$

6. A landscape architect wishes to enclose a rectangular garden on one side by a brick wall costing \$30 per foot and on the other three sides with a metal fence costing \$10 per foot. The area of the garden is to be 800ft². What are the dimensions of the garden that minimize the cost of the fencing?



Cost: $C = 30y + 20x + 10y = 40y + 20x$

Area: $A = xy = 800$

$$y = \frac{800}{x}$$

$$C = \frac{40 \cdot 800}{x} + 20x$$

$$C' = -\frac{32000}{x^2} + 20$$

$$C'' = \frac{2 \cdot 32000}{x^3} > 0 \text{ on } (0, \infty)$$

$C' = 0$: $\dfrac{16000}{x^2} = 1$

$$x = \sqrt{16000} = 4 \cdot 10 \cdot \sqrt{10} = 40 \cdot \sqrt{10}$$

$$y = \frac{800}{x} = \frac{800}{40\sqrt{10}} = 2\sqrt{10}$$

7.

a. State the Mean Value Theorem and draw a picture to illustrate it.

If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), then there is c in (a,b) where

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

b. Suppose $f(x)$ is continuous on $[-1,1]$ and has a derivative at each x in $(-1,1)$. If $f(-1) = 7$ and $f(1) = 5$, what does the Mean Value Theorem let you conclude?

There is a c in $(-1,1)$ where

$$f'(c) = \frac{5-7}{1-(-1)} = \frac{-2}{2} = -1$$