1. State, formally, the definition of the derivative of a function $f(x)$ at $x = a$.

$$\lim_{a \to x} \frac{f(x)-f(a)}{x-a} \quad \text{or} \quad \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

2. Let $f(x) = 5x^2 - 3x$.

1. Use the definition to find the derivative of $f(x)$.

$$f'(x) = \lim_{a \to x} \frac{f(x)-f(a)}{x-a} = \lim_{a \to x} \frac{5x^2-3x-(5a^2-3a)}{x-a}$$

$$= \lim_{a \to x} \frac{5(x^2-a^2)-3(x-a)}{x-a}$$

$$= \lim_{a \to x} \frac{5(x-a)(x+a)-3(x-a)}{x-a}$$

$$= \lim_{a \to x} 5(x+a)-3 = 5(a+a)-3$$

$$= \boxed{10a-3}$$

2. Find the slope of the tangent line to $f(x)$ when $x = -3$.

$$f'(-3) = -33$$

3. Write the equation of the line tangent to $f(x)$ when $x = -3$.

$$y = f(-3) + f'(-3)(x-(-3))$$

$$= 54 - 33(x+3)$$

3. Suppose N represents the number of people in the United States who travel by car to another state for a vacation this Memorial Day weekend when the average price of gasoline is p dollars per gallon.

1. What are the units of dN/dp?

   people / dollar

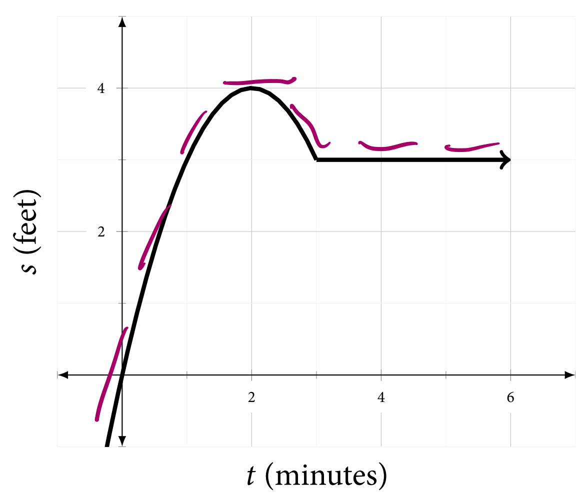
2. In the context of the problem, interpret $\frac{dN}{dp}$.

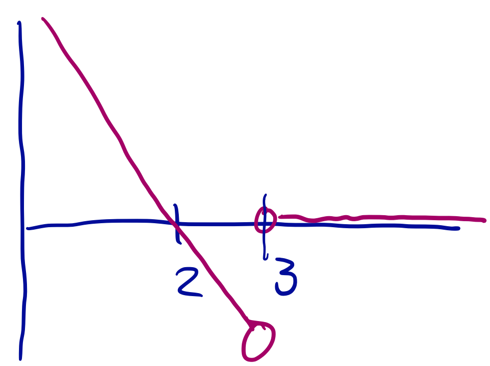
   This is the rate at which the number of travellers changes as the price of gas increases.

3. Would you expect dN/dp to be positive or negative? Explain your answer.

   Negative. The number of travelers should decrease as the price of gas goes up.

4. The graph of $f(x)$ is sketched below. On a separate set of axes, give a rough sketch $f'(x)$.





5. Find the domain of each function.

1. $f(x) = \sqrt{x^2 - x - 6}$

   Need $x^2 - x - 6 \geq 0$

   $(x+2)(x-3) \geq 0$

   so $x \leq -2$ or $x \geq 3$

   $(-\infty, -2] \cup [3, \infty)$

2. $g(t) = \ln(t+6)$

   $t + 6 > 0$

   $t > -6$

   $(-6, \infty)$

6. State the definition of "The function $f(x)$ is continuous at $x = a$".

$$\lim_{x \to a} f(x) = f(a)$$

7. Suppose

$$f(x) = \begin{cases} -\frac{2}{x} & x < 2 \\ \frac{x}{x-3} & x \geq 2 \end{cases}$$

Is $f(x)$ continuous at $x = 0$? At $x = 2$? Justify your answers using the definition of continuity.

At 0? No. The function isn't defined there.

At $x=2$? No. $\lim_{x \to 2^-} \frac{-2}{x} = -1$, $\lim_{x \to 2^+} \frac{x}{x-3} = \frac{2}{-1} = -2$.

Since $-1 \neq -2$, $\lim_{x \to 2} f(x)$ does not exist, much less equal $f(2)$.

8. Find the limit or show that it does not exist. *Make sure you are writing your mathematics correctly and clearly.*

1. $\lim_{x \to \infty} \frac{10^x - 1}{3 - 10^x}$ $= \lim_{x \to \infty} \frac{1 - 10^{-x}}{3 \cdot 10^{-x} - 1} = \frac{1 - 0}{0 - 1} = -1$

2. $\lim_{x \to \infty} \frac{\sqrt[3]{8x^3 + 1}}{2 - 5x}$ $= \lim_{x \to \infty} \frac{x \sqrt[3]{8 + 1/x^3}}{2 - 5x}$

$$= \lim_{x \to \infty} \frac{\sqrt[3]{8 - 1/x^3}}{2/x - 5}$$

$$= \frac{\sqrt[3]{8}}{-5} = \frac{-2}{5}$$

6. State the definition of "The function $f(x)$ is continuous at $x = a$".

7. Suppose

$$f(x) = \begin{cases} -\frac{2}{x} & x < 2 \\ \frac{x}{x-3} & x \geq 2 \end{cases}$$

Is $f(x)$ continuous at $x = 0$? At $x = 2$? Justify your answers using the definition of continuity.

8. Find the limit or show that it does not exist. *Make sure you are writing your mathematics correctly and clearly.*

(a) $\displaystyle\lim_{x\to\infty} \frac{10^x - 1}{3 - 10^x}$

(b) $\displaystyle\lim_{x\to\infty} \frac{\sqrt[3]{8x^3 + 1}}{2 - 5x}$

(c) $\displaystyle\lim_{r\to 16^-} \frac{\overset{+}{\sqrt{r}}}{\underset{-}{(r - 16)^3}} = -\infty$

as $r \to 16^-$, $\sqrt{r} = 4$ and $(r-16)^3 \to 0^-$

(d) $\displaystyle\lim_{x\to -3} \frac{x^2 - 9}{x^2 + 2x - 3} = \lim_{x\to -3} \frac{(x+3)(x-3)}{(x+3)(x-1)} = \lim_{x\to -3} \frac{x-3}{x-1} = \frac{-6}{-4} = \frac{3}{2}$

9. Consider a function with vertical asymptotes at $x = -1$ and $x = 3$ and a horizontal asymptote at $y = 4/3$.

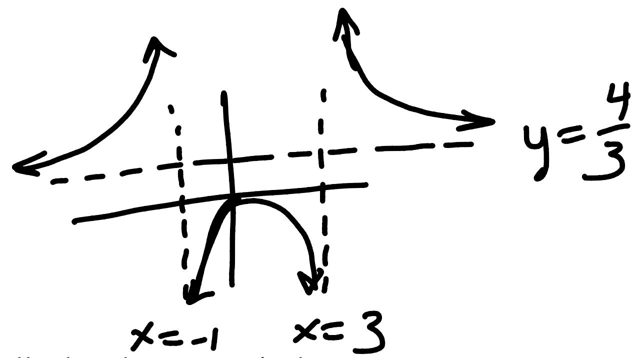
(a) Write a formula for such a function.

$$f(x) = \frac{4x^2}{3(x+1)(x-3)}$$

$-3+2$

(b) Sketch the graph of the function.

roughly



(c) Use limits to demonstrate that your function really does have a vertical asymptote at $x = -1$

$$\lim_{x \to -1^+} \frac{4x^2}{3(x+1)(x-3)} = -\infty$$

$x \to -1^+$, $4x^2 \to 4$

and $x+1 \to 0^+$, $x-3 \to -4$

(d) Use limits to demonstrate that your function really does have a horizontal asymptote at $y = 4/3$.

$$\lim_{x \to \infty} \frac{4x^2}{3x^2-6x-9} = \lim_{x \to \infty} \frac{4}{3 - \frac{6}{x} - \frac{9}{x^2}} = \frac{4}{3}$$

11. Solve for x.

1. $e^{x-3} + 2 = 6$

$e^{x-3} = 4$

$x-3 = \ln(4)$

$x = 3 + \ln(4)$

2. $\ln(x + 5) - 3 = 7$

$\ln(x+5) = 10$

$x+5 = e^{10}$

$x = e^{10} - 5$

3. $\ln x + \ln(x - 1) = 0$

$\ln(x(x-1)) = 0$

$x(x-1) = e^0 = 1$

$x^2 - x - 1 = 0$

$x = \frac{1 \pm \sqrt{1+4}}{2}$

$= \frac{1 \pm \sqrt{5}}{2}$

But $\frac{1-\sqrt{5}}{2}$ is not in the domain.

4. $\cos(8x) = 0$

$8x = \frac{\pi}{2} + k\pi$

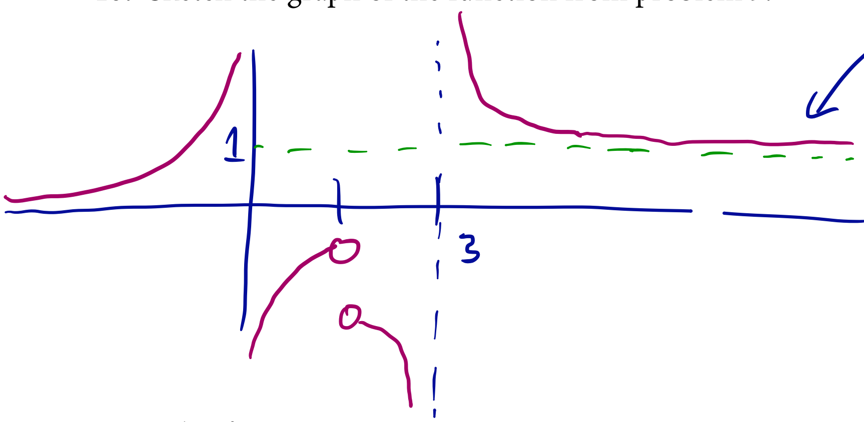
$x = \frac{\pi}{16} + k\frac{\pi}{8}$

$k \in \mathbb{Z}$

9. Write the formula for a function with vertical asymptotes at $x = -1$ and $x = 3$ and a horizontal asymptote at $y = 4/3$.

$$f(x) = \frac{1}{(x+1)(x-3)} + \frac{4}{3}$$

10. Sketch the graph of the function from problem 7.



$\lim_{x\to\infty} f(x) = 1$

Note:

$$\frac{x}{x-3} = \frac{x-3}{x-3} + \frac{3}{x-3} = 1 + \frac{3}{x-3}$$

11. Solve for x.

1. $e^{x-3} + 2 = 6$

$e^{x-3} = 4$

$x-3 = \ln(4)$

$x = 3 + \ln(4)$

2. $\ln(x+5) - 3 = 7$

$\ln(x+5) = 10$

$x+5 = e^{10}$

$x = e^{10} - 5$

3. $\ln x + \ln(x-1) = 0$

$\ln(x(x-1)) = 0$

$x(x-1) = e^0 = 1$

$x^2 - x - 1 = 0$ $\quad x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$

But $\frac{1-\sqrt{5}}{2}$ is not in the domain.

4. $\cos(8x) = 0$

$8x = \frac{\pi}{2} + k\pi$

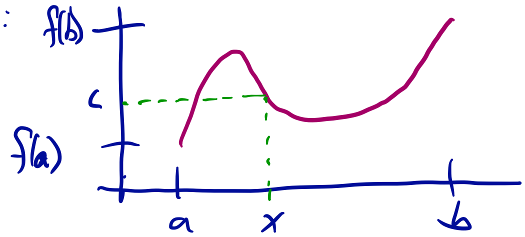
$x = \frac{\pi}{16} + k\frac{\pi}{8}$

$k \in \mathbb{Z}$

12.

1. What does the Intermediate Value Theorem say? You may want to include a picture with your explanation.

If $f(x)$ is continuous on $[a,b]$ and c is a number between $f(a)$ and $f(b)$ then there is x in $[a,b]$ with $f(x)=c$:



2. Use the Intermediate Value Theorem to show $\ln x = x - 5$ has a solution. (Hint: Show there is a solution in the interval $[1, e^5]$.)

Let $f(x) = \ln(x) - x + 5$. Notice $f(x)$ is continuous on $(0,\infty)$ and so also on $[1, e^5]$. Moreover,

$f(1) = 0 - 1 - 5 = -6 < 0$ and

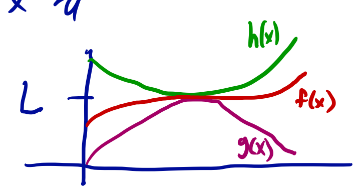
$f(e^5) = \ln(e^5) + e^5 - 5 = e^5 > 0$.

So there is x in $[1, e^5]$ with $f(x) = 0$.

13.

1. What does the Squeeze Theorem say? You may want to include a picture with your explanation.

If $g(x) \le f(x) \le h(x)$ near $x=a$, but maybe not at $x=a$, and if $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} f(x) = L$ also.



14. Use the Squeeze Theorem to show $\lim\limits_{x\to\infty} \dfrac{\cos(2x)}{3x^2} = 0$.

Since $-1 \le \cos(2x) \le 1$, $\dfrac{-1}{3x^2} \le \dfrac{\cos(2x)}{3x^2} \le \dfrac{1}{3x^2}$.

Since $\lim\limits_{x\to\infty} -\dfrac{1}{3x^2} = \lim\limits_{x\to\infty} \dfrac{1}{3x^2} = 0$, $\lim\limits_{x\to\infty} \dfrac{\cos(2x)}{3x^2} = 0$.

15. Sketch each of the functions below. Label all x- and y-intercepts and asymptotes. State, in interval notation, the domain and range of each function next to its graph.

1. $y = 6 - x^4$
2. $y = \sin(2x)$
3. $y = \tan x$
4. $y = \tan^{-1} x$
5. $y = e^{x-1} + 2$
6. $y = \ln x$
7. $y = -2/(x+3)$
8. $y = \sqrt{x+5}$