1. Find the derivative of

(a) $y = (3x - x^5)^{2/3}(x - \tan(x))^5$.

Change the problem

$$\ln y = \ln\left[(3x-x^5)^{2/3}(x-\tan(x))^5\right]$$

$$\ln y = \frac{2}{3}\ln(3x-x^5) + 5\ln(x-\tan x)$$

take derivative implicitly

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{2}{3}\left(\frac{3-5x^4}{3x-x^5}\right) + 5\left(\frac{1-\sec^2 x}{x-\tan x}\right)$$

$$dy/dx = y\left(\frac{2(3-5x^4)}{3(3x-x^5)} + \frac{5(1-\sec^2 x)}{x-\tan x}\right)$$

Replace y

$$\frac{dy}{dx} = \boxed{(3x-x^5)^{2/3}(x-\tan x)^5}\left(\frac{2(3-5x^4)}{3(3x-x^5)} + \frac{5(1-\sec^2 x)}{x-\tan x}\right)$$

(b) Find the derivative of $y = (\sin(x))^x$.

Take natural log of both sides

$$\ln y = \ln\left((\sin x)^x\right)$$

$$\ln y = x\ln(\sin x)$$

Take derivative implicitly

$$\left(\frac{1}{y}\right)\left(\frac{dy}{dx}\right) = \ln(\sin x) + x\left(\frac{1}{\sin x}\right)(\cos x)$$

$$\frac{dy}{dx} = y\left(\ln(\sin x) + \frac{x\cos x}{\sin x}\right)$$

Replace y

$$\frac{dy}{dx} = [\sin(x)]^x\left(\ln(\sin x) + x\cot x\right)$$

2. A ball is tossed straight up into the air. It has a velocity at time $t = 0$ seconds of 5 meters per second. It undergoes a constant acceleration due to gravity of -9.8 meters per second per second, $\mathrm{m/s^2}$. The height of the ball can be written in the form

$$h(t) = at + bt^2 = 5t - 4.9t^2$$

where h is measured in meters, time is measured in seconds, and a and b are certain constants.

(a) Determine the values for the constants.

$h'(t) = a + 2bt$ $\quad h''(t) = 2b = -9.8$; So $b = -4.9$ ✓

$v = h'(0) = a = 5$ ✓

(b) What is the height of the ball at time $t = 0$? At $t = 1$?

$h(0) = 0\text{ m}$, $\quad h(1) = 5 - 4.9 = 0.1\text{ m}$

(c) At what times is the ball at height 0?

$0 = 5t - 4.9t^2 = t(5 - 4.9t)$

$t = 0$ or $t = 5/4.9 = 1.02$ sec

(d) What is the average velocity of the ball over the time interval $[0.2, 0.21]$?

$$\frac{h(0.21) - h(0.2)}{0.21 - 0.2} = 2.991\text{ m/s}$$

(e) What is the average velocity of the ball over the time interval $[0.2, 0.201]$?

$$\frac{h(0.201) - h(0.2)}{0.201 - 0.2} = 3.0351\text{ m/s}$$

It fits.

(f) What is the instantaneous velocity of the ball at time $t = 0.2$??

$h'(t) = 5 - 9.8t \quad h'(0.2) = 5 - 9.8(0.2) = 3.04\text{ m/s}$

(g) At what time t is the ball motionless?

$h'(t) = 0 = 5 - 9.8t$. So $t = 5/9.8 = 0.5102\text{ s}$

What is happening here...

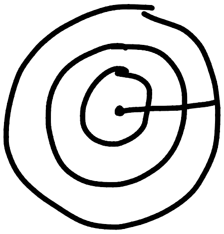
(h) What is the velocity of the ball at time $t = 0$? At $t = 0.1$? At $t = 1$?

$\boxed{h(0) = 5\text{ m/s}}$ $\quad h(1) = -4.8\text{ m/s}$

$h(0.1) = 4.02\text{ m/s}$

Find the area inside ripple when $t=1, t=2$.

3. A stone is thrown in a pond and a circular ripple travels outward at a speed of 60 cm/s. Determine the rate of change of area inside the ripple at time $t = 1$ second and at time $t = 2$ seconds.



$r(t) = 60t$

r - metres, t - seconds

$A(t) = \pi(60t)^2 = 3600\pi t^2$

$A(1) = 3600\pi \text{ cm}^2$

$A(2) = 4(3600\pi) \text{ cm}^2$

$= 14400\pi \text{ cm}^2$

$A'(t) = 7200\pi t$

$A'(1) = 7200\pi \text{ cm}^2/\text{s}$

$A'(2) = 14400\pi \text{ cm}^2/\text{s}$

4. A population of bacteria starts at 500 cells and doubles every 30 minutes. Find a function $P(t)$ that describes this situation. Then compute the rate of change of the bacteria population at time $t = 60$ minutes.

$P_0 \xrightarrow{30} 2P_0 \xrightarrow{30} 4P_0 \xrightarrow{30} 8P_0$

30 min   60 m   90 min

P - pop of bacteria, t - seconds

$\boxed{P(t) = 500\, 2^{t/30}}$

$P'(t) = (500)(\ln 2)\left(2^{t/30}\right) \cdot \frac{1}{30} = \left(\frac{50 \ln 2}{3}\right) 2^{t/30}$

$P'(60) = \frac{50 \ln 2}{3} 2^2 = 46$ bacteria/min

5. A population of caribou is growing, and its population is

$$P(t) = 4000\frac{3e^{t/5}}{1+2e^{t/5}}.$$

(a) What is the population at time $t = 0$?

$$P(0) = 4000\left(\frac{3e^0}{1+2e^0}\right) = 4000\left(\frac{3}{3}\right) = 4000 \text{ caribou}$$

(b) Determine the rate of change of the population at any time t.

$$P'(t) = 4000\left[\frac{(1+2e^{t/5})(3e^{t/5}(1/5)) - (3e^{t/5})(\frac{2}{5}e^{t/5})}{(1+2e^{t/5})^2}\right]$$

$$= 4000\left[\frac{\frac{3}{5}e^{t/5}}{(1+2e^{t/5})^2}\right] = \frac{2400e^{t/5}}{(1+2e^{t/5})^2}$$

(c) Determine the rate of change of the population at time $t = 0$ years.

$$P'(0) = \frac{2400}{3^2} = 266 \text{ caribou/yr}$$

(d) Determine the long term population.

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} 4000\left(\frac{3e^{t/5}}{1+2e^{t/5}}\right) = \lim_{t\to\infty} \frac{4000\cdot 3}{\frac{1}{e^{t/5}}+2}$$

$$= 6000 \text{ caribou}$$