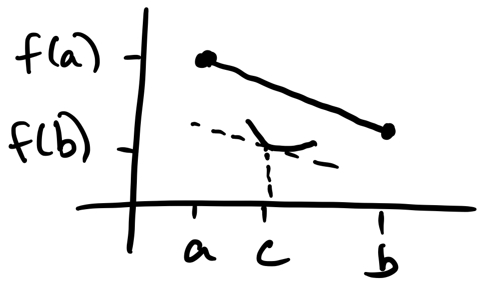
4-3

Mean Value Theorem. If f is a continuous function on an interval $[a, b]$ that has a derivative at every point in (a, b), then there is a point c in (a, b) where

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

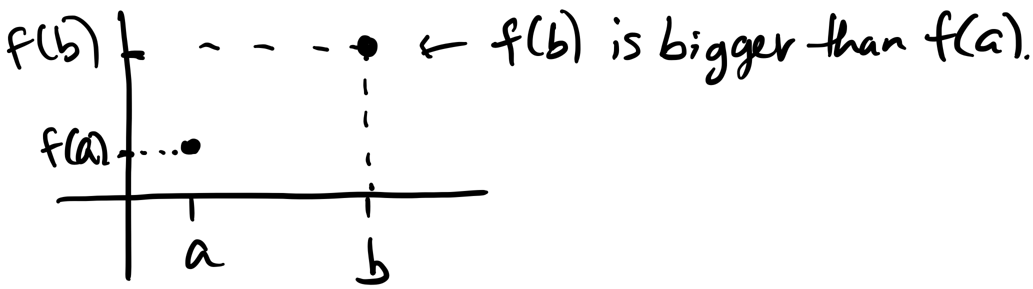
1. Suppose f is a continuous function on $[a, b]$ that has a derivative at every point of (a, b). Suppose also that $f(b) \leq f(a)$. What can you conclude from the Mean Value Theorem?

There is some c in (a,b) where $f'(c) \leq 0$

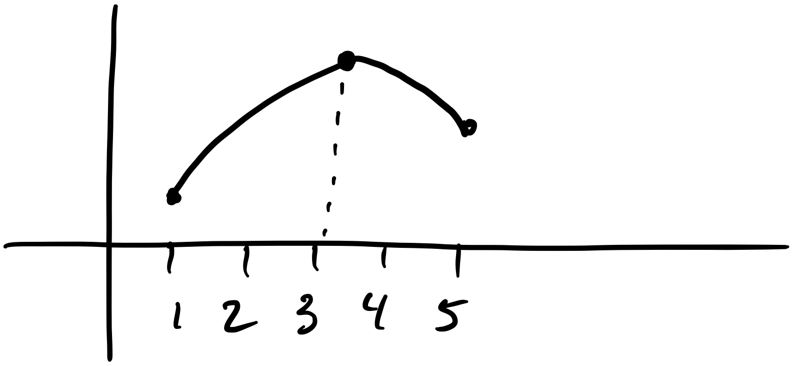


2. Suppose f is a continuous function on $[a, b]$ that has a derivative at every point of (a, b), and that $f'(x) > 0$ for each x in (a, b). Thinking about your answer to problem 1, what can you conclude about $f(a)$ and $f(b)$?

If $f' > 0$ everywhere, then $f(b) > f(a)$.



3. A function is said to be **increasing** on an interval (a, b) if whenever x and z are in the interval and $x < z$, then $f(x) < f(z)$. It is **decreasing** if whenever x and z are in the interval and $x < z$, then $f(x) > f(z)$ Sketch an example of a function that is increasing on $(1, 3)$ and decreasing on $(3, 5)$.



Increasing/Decreasing Test

Your answer to problem 2 implies the first item below; the second is justified by a similar argument.

- If $f'(x) > 0$ on an interval (a, b) then f is increasing on the interval.
- If $f'(x) < 0$ on an interval (a, b) then f is decreasing on the interval.

4. Use the increasing/decreasing test to find intervals where

$$f(x) = \frac{2}{3}x^3 + x^2 - 12x + 7$$

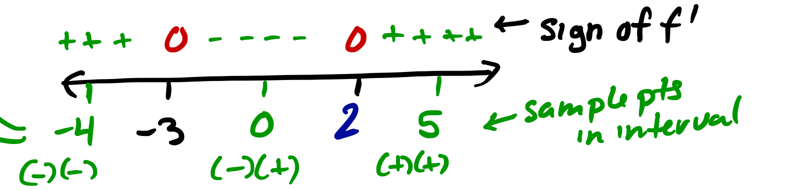
is increasing and intervals where it is decreasing.

I need to find where $f'>0$ and $f'<0$.

$f'(x) = 2x^2 - 2x - 12$

$= 2(x^2 - x - 6)$

$= 2(x-2)(x+3)$



$f' = 0$ when $x = 2, -3$.

answer: f is increasing on $(-\infty, -3) \cup (2, \infty)$ and decreasing on $(-3, 2)$.

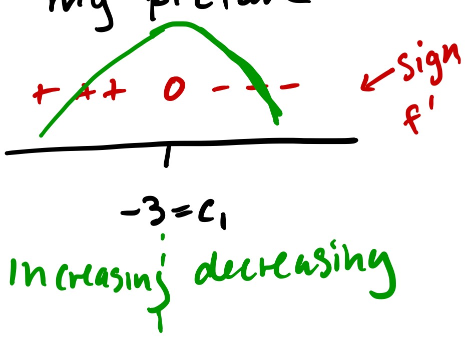
5. Find the critical points of the function $f(x) = \frac{2}{3}x^3 + x^2 - 12x + 7$ from the previous problem. There should be two, c_1 and c_2 with $c_1 < c_2$. Just pay attention to c_1.

   (a) Just to the left of c_1 is the function increasing or decreasing? increasing

   (b) Just to the right of c_1 is the function increasing or decreasing? decreasing

   (c) Now decide intuitively, based on these two observations, if f has a local min, local max, or neither at c_1.

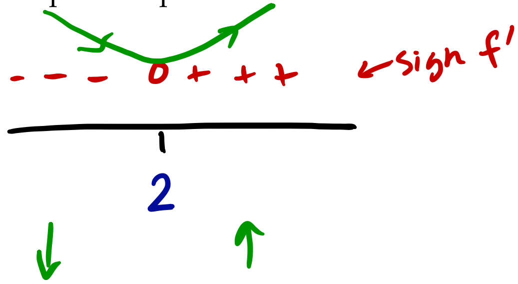
my picture



answer: f has a local max at $x=-3$

That is $f(-3) = 34$ is a local max

6. Repeat the previous exercise for the other critical point c_2.



answer: f has a local min at $x=2$.

That is $f(2) = -7.\overline{6}$ is a local min.

You have just sketched the argument that justifies the following:

First Derivative Test
Suppose f is a function with a derivative on (a, b), and if c is a point in the interval with $f'(c) = 0$.

- If $f'(x) > 0$ for x just to the left of c and $f'(x) < 0$ for x just to the right of c, then f has a local max at c.

  ↑ + + + 0 − − − ↓

  c

- If $f'(x) < 0$ for x just to the left of c and $f'(x) > 0$ for x just to the right of c, then f has a local min at c.

  ↓ − − − − 0 + + + ↑

  c

7. The function $f(x) = xe^x$ has exactly one critical point. Find it, and then use the First Derivative Test to determine if a local minimum or local maximum occurs there.

$f'(x) = 1 \cdot e^x + x \cdot e^x$

$= e^x(1+x)$

c.p. $x = -1$.

Check on either side of $x = -1$

− 0 + ← sign of f'

−2 −1 0 ← pick pts on either side

picture

↓ x ↑

answer

local min at $x = -1$

local min is $f(-1) = \frac{-1}{e}$

8. Consider the function $f(x) = \frac{2}{3}x^3 + x^2 - 12x + 7$. Find intervals such that the **derivative** of $f(x)$ is increasing or decreasing.

$f'(x) = 2x^2 - 2x - 12$ ← where is THIS increasing/decreasing.

So $f''(x) = 4x - 2 = 2(2x-1)$

$x = \frac{1}{2}$ ← crit. pt for f'.

− − 0 + + + ← sign of f''

−3 0 $x=\frac{1}{2}$ 1 ← sample points

answer

f' is increasing on $(\frac{1}{2}, \infty)$

and f' is decreasing on $(-\infty, \frac{1}{2})$.

9. Earlier you computed that $f'(-3) = 0$. Is f' increasing near $x = -3$ or decreasing near $x = -3$? because
Which of the following two scenarios must we have?

$\boxed{f'(x) \quad + \quad -3 \quad -}$

$f'(x) \quad - \quad -3 \quad +$

positive #'s decrease to negative #'s

negative #'s increase to positive numbers

You have just sketched out justification for the following.

Second Derivative Test
Suppose f is a function with a continuous second derivative on (a, b), and that c is a point in the interval with $f'(c) = 0$.

increasing f' means $-$ to $+$ means $\searrow\nearrow$

- If $f''(c) > 0$ then f has a local min at c.

decreasing f' means $+$ to $-$ means $\nearrow\searrow$

- If $f''(c) < 0$ then f has a local max at c.

10. Use the Second Derivative Test to determine if $f(x) = xe^x$ has a local min/max at its only critical point.

$f'(x) = e^x(1+x)$

c.p. $x=-1$

$f''(x) = e^x(1+x) + 1\cdot e^x$
$= e^x(x+2)$

$f''(-1) = e^{-1}(-1+2) = e^{-1} > 0$

So $f(-1) = \frac{-1}{e}$ is a local min.

11. Consider the function $f(x) = x^3$. Verify that $f'(0) = 0$. Then decide what the Second Derivative Test has to say, if anything, about whether a local min/max occurs at $x = 0$.

$f'(x) = 3x^2$

$f''(x) = 6x$

$f''(0) = 6\cdot 0 = 0$ ← neither positive nor negative.

The 2^{rd} Der Test is a bust. No info.

12. Decide what the First Derivative Test has to say, if anything, about whether a local min/max occurs at $x = 0$ for $f(x) = x^3$.

$f'(x) = 3x^2$
check sign f' on either side of $x=0$.

+ 0 + ← sign f'

$x=0$

← increasing always

The First DerTest tells us neither max nor min occurs here.