SECTION 5-3: THE FUNDAMENTAL THEOREM OF CALCULUS

Example 1: If f is the function whose graph is shown and $g(x) = \int_0^x f(t)dt$, find the values of $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$, $g(5)$, and $g(6)$. Then, sketch a rough graph of g.

(a) $g(0) =$ 0

(b) $g(1) =$ 1

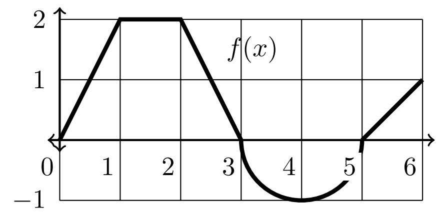
(c) $g(2) =$ 3

(d) $g(3) =$ 4

(e) $g(4) =$ 3.21

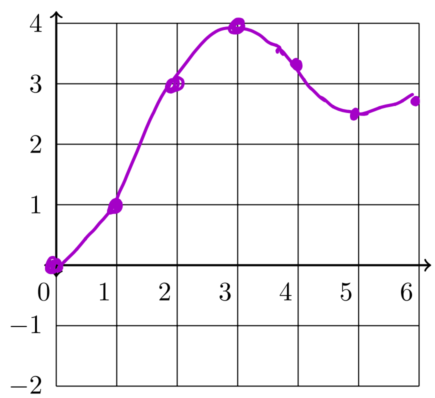
(f) $g(5) =$ 2.43

(g) $g(6) =$ 2.93



area of quarter-circle is $\frac{\pi}{4}$ $\approx .78$

Sketch of $g(x)$



(i) Where is $g(x)$ increasing? [0, 3] and [5, 6]

(ii) Describe f when $g(x)$ is increasing. positive

(iii) Where is $g(x)$ decreasing? [3, 5]

(iv) Describe f when $g(x)$ is decreasing. negative

(v) Where does $g(x)$ have a local maximum? $x = 3$

(vi) Describe f when $g(x)$ has a local max. zero and goes $+ \to -$

(vii) Where does $g(x)$ have a local minimum? $x = 5$

(viii) Describe f when $g(x)$ has a local min. zero and goes $- \to +$

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, the function g defined by

$$g(x) = \int_a^x f(t)\,dt \quad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b) and $g'(x) = f(x)$.

Example 2: Find the derivative of $g(x) = \displaystyle\int_2^x t^2\,dt$.

By FTC I, $g'(x) = x^2$.

Example 3: The Fresnel function $S(x) = \int_0^x \sin(\pi t^2/2)dt$ first appeared in Fresnel's theory of the diffraction of light waves. Recently it was be applied to the design of highways. Find the derivative of the Fresnel function.

By FTC 1, $S'(x) = \sin\left(\frac{\pi x^2}{2}\right)$.

Example 4: Find the derivative of the following functions. (Hint: we need to use the chain rule! For part (a), let $u = x^4$...)

(a) $g(x) = \int_1^{x^4} \sec t dt$

$g'(u) = \sec(u)$

so $g'(x) = \sec u \frac{du}{dx}$

$= \sec(x^4)(4x^3)$

(b) $g(x) = \int_{2x+1}^2 \sqrt{t} dt = -\int_2^{2x+1} \sqrt{t}\, dt$

$= -\sqrt{2x+1}\,(2)$

Example 5: Find the derivative of $g(x) = \int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^4}} dt$. (Hint: we only know the derivative of $\int_a^x f(t)\, dt$, so you need to break this into pieces...)

$g(x) = \int_0^{x^2} \frac{1}{\sqrt{2+t^4}} dt + \int_{\tan(x)}^{0} \frac{1}{\sqrt{2+t^4}} dt = \int_0^{x^2} \frac{1}{\sqrt{2+t^4}} dt - \int_0^{\tan x} \frac{1}{\sqrt{2+t^4}} dt$, so

$g'(x) = \frac{1}{\sqrt{2+(x^2)^4}}(2x) - \frac{1}{\sqrt{2+(\tan x)^4}}(\sec^2(x)) = \frac{2x}{\sqrt{2+x^8}} - \frac{\sec^2(x)}{\sqrt{2+(\tan(x))^4}}$

The Fundamental Theorem of Calculus (Part 2) If f is continuous on $[a,b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

where F is **any antiderivative** of f, that is, is a function such that $F' = f$. To determine $F(b) - F(a)$ we write $F(x)\Big|_a^b = F(b) - F(a)$

Example 6: Evaluate the following integrals.

(a) $\int_0^1 x^2\, dx$

$= \frac{x^3}{3}\Big|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$

(we did this on a previous worksheet!)

(b) $\int_{\cancel{0}\,1}^4 (1 + 3y - y^2)\, dy$

$= \left(y + \frac{3y^2}{2} - \frac{y^3}{3}\right)\Big|_1^4$

$= \left(4 + \frac{3 \cdot 16}{2} - \frac{64}{3}\right) - \left(1 + \frac{3}{2} - \frac{1}{3}\right)$

$= 4 + 24 - \frac{64}{3} - 1 - \frac{3}{2} + \frac{1}{3}$

$= 27 - 21 - 3/2 = \frac{9}{2}$

To compute integrals effectively you **must** have your basic antidifferentiation formulas down. You should know that antiderivatives to the following functions. Note, we are using the $\int$ symbol to mean "find the antiderivative" of the function right after the symbol.

Antiderivatives of Common Functions:

- $\int x^n \, dx = \frac{x^{n+1}}{n+1} \quad n \neq -1$
- $\int \sin x \, dx = -\cos(x)$
- $\int \cos x \, dx = \sin(x)$
- $\int \sec^2 x \, dx = \tan(x)$
- $\int \sec x \tan x \, dx = \sec(x)$
- $\int \csc^2 x \, dx = -\cot(x)$
- $\int \csc x \cot x \, dx = -\csc(x)$
- $\int e^x \, dx = e^x$
- $\int a^x \, dx = \frac{a^x}{\ln(a)}$
- $\int \frac{1}{1+x^2} \, dx = \arctan(x)$
- $\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin(x)$
- $\int \frac{1}{x} \, dx = \ln|x|$

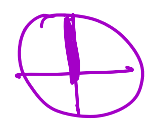
Example 7: Evaluate the following integrals.

(a) $\int_2^5 \frac{3}{x} \, dx$

$= 3\ln|x| \Big|_2^5 = 3(\ln(5) - \ln(2))$

$= 3\ln(5/2)$

(b) $\int_0^{\pi/2} \cos x \, dx$



$= \sin(x) \Big|_0^{\pi/2}$

$= \sin(\frac{\pi}{2}) - \sin(0)$

$= 1 - 0$

$= 1$

Example 8: Evaluate the following integrals.

(a) $\int_1^8 \sqrt[3]{x} \, dx$

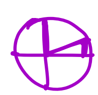
$= \int_1^8 x^{1/3} \, dx = \frac{x^{1/3+1}}{1/3+1} \Big|_1^8$

$= \frac{3x^{4/3}}{4} \Big|_1^8 = \frac{3 \cdot 2^4}{4} - \frac{3 \cdot 1^4}{4}$

$= 3(4) - \frac{3}{4} = \frac{48-3}{4}$

$= \frac{45}{4}$

(b) $\int_{\pi/6}^{\pi/2} \csc x \cot x \, dx$



$= -\csc(x) \Big|_{\pi/6}^{\pi/2}$

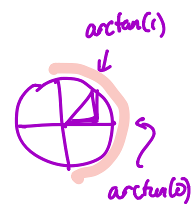
$= \frac{-1}{\sin(x)} \Big|_{\pi/6}^{\pi/2}$

$= \frac{-1}{\sin(\frac{\pi}{2})} + \frac{1}{\sin(\pi/6)}$

$= -1 + \frac{1}{1/2}$

$= 1$

(c) $\int_0^1 \frac{9}{1+x^2} \, dx$



$= 9 \int_0^1 \frac{1}{1+x^2} \, dx$

$= 9 \arctan \Big|_0^1$

$= 9\arctan(1) - 9\arctan(0)$

$= 9(\frac{\pi}{4}) - 0$

$= \frac{9\pi}{4}$

Example 9: We do not have any product or quotient rules for antidifferentiation. To evaluate an integral that is expressed as a product or quotient you must try to manipulate the integrand (the stuff inside the $\int$ sign) to look like something you know how to anti-differentiate. The following integrals are examples of this. Evaluate the following integrals.

(a) $\displaystyle\int_1^3 \frac{x^3+3x^6}{x^4}\,dx$

$= \int_1^3 \frac{1}{x}\,dx + 3\int_1^3 x^2\,dx$

$= \ln|x|\Big|_1^3 + 3\,\frac{x^3}{3}\Big|_1^3$

$= \ln(3) - \ln(1) + 3^3 - 1^3$

$= \ln(3) + 26$

(b) $\displaystyle\int_0^1 x(3+\sqrt{x})\,dx$

Note $x = \sqrt{x^2}$ when $x > 0$

$= \int_0^1 3x + x\sqrt{x}\,dx$

$= \int_0^1 3x + x^{3/2}\,dx$

$= \frac{3x^2}{2} + \frac{x^{5/2}}{5/2}\Big|_0^1$

$= \frac{3}{2}x^2 + \frac{2x^{5/2}}{5}\Big|_0^1$

$= \frac{3}{2}(1)^2 + \frac{2}{5}(1)^{5/2} - 0$

$= \frac{3}{2} + \frac{2}{5} = \frac{15+4}{10} = \frac{19}{10}$

Example 10: Evaluate the following integrals.

(a) $\displaystyle\int_0^2 (5^x + x^5)\,dx$

$= \frac{5^x}{\ln(5)} + \frac{x^6}{6}\Big|_0^2$

$= \frac{5^2}{\ln(5)} + \frac{2^6}{6} - \frac{5^0}{\ln(5)} - 0$

$= \frac{24}{\ln(5)} + \frac{64}{6} = \frac{24}{\ln(5)} + \frac{32}{3}$

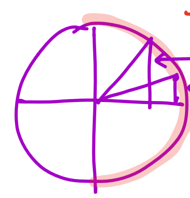
(b) $\displaystyle\int_{1/2}^{\sqrt{2}/2} \frac{1}{\sqrt{1-x^2}}\,dx$

$= \arcsin(x)\Big|_{1/2}^{\sqrt{2}/2}$

$= \arcsin\left(\frac{\sqrt{2}}{2}\right) - \arcsin(1/2)$

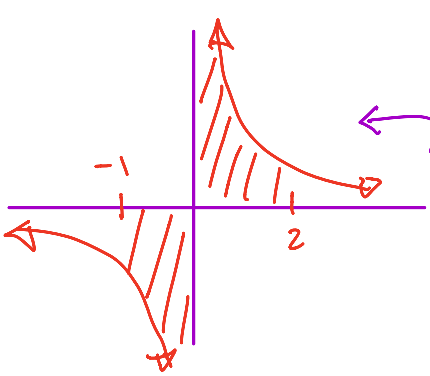
$= \frac{\pi}{4} - \frac{\pi}{6} = \frac{3\pi}{12} - \frac{2\pi}{12}$

$= \pi/12.$



Example 11: What is wrong with the following calculation?

$$\int_{-1}^{3} \frac{1}{x^2}\,dx = \frac{x^{-1}}{-1}\Big]_{-1}^{3} = -\frac{1}{3} - 1 = -\frac{4}{3}$$



$\frac{1}{x^2}$ is not continuous on $[-1, 3]$.

FTC 2 does not apply. We will need a new technique in Calc 2 to compute this limit!