Definite Integrals and Areas "under" Curves

1. Estimate the area under $f(x) = x^2 - 2x$ on ~~$[1,4]$~~ $[1,3]$ with $n = 6$ $n=4$ using the

   (a) Right-hand endpoints

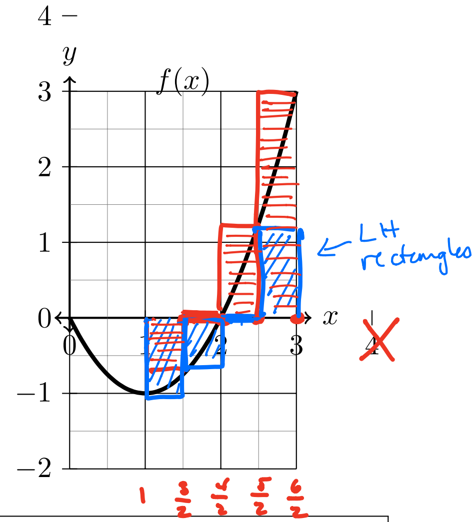
   Width $= \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$

   Area $= \frac{1}{2}\left(\left(\frac{3}{2}\right)^2 + 2\left(\frac{3}{2}\right)\right) + \frac{1}{2}\left(\left(\frac{4}{2}\right)^2 + 2\left(\frac{4}{2}\right)\right) + \frac{1}{2}\left(\left(\frac{5}{2}\right)^2 + 2\left(\frac{5}{2}\right)\right) + \frac{1}{2}\left(\left(\frac{6}{2}\right)^2 + 2\left(\frac{6}{2}\right)\right)$

   (b) Left-hand endpoints

   Width $= \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$

   Area $= \frac{1}{2}\left((1)^2 + 2(1)\right) + \frac{1}{2}\left(\left(\frac{3}{2}\right)^2 + 2\left(\frac{3}{2}\right)\right) + \frac{1}{2}\left(\left(\frac{4}{2}\right)^2 + 2\left(\frac{4}{2}\right)\right) + \frac{1}{2}\left(\left(\frac{5}{2}\right)^2 + 2\left(\frac{5}{2}\right)\right)$



Definition of a Definite Integral If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b-a)/n$. We let $x_0 = (a), x_1, x_2, \cdots, x_n = (b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \cdots, x_n^*$ be **sample points**[1] in these subintervals, so x_i^* lies in the i-th subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

provided this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

[1] For example, we could choose our sample points to be right-hand endpoints, left-hand endpoints, midpoints, a combination of these, or any other sample points in the interval that we choose!

2. Consider again $f(x) = x^2 - 2x$ on the interval $[1, 3]$. Suppose that we are dividing the interval $[1, 3]$ into n subintervals. (Think about your answers to #1.)

   (a) What is the length of each subinterval? $\frac{3-1}{n} = \frac{2}{n}$

   (b) What is the right-hand endpoint of the first subinterval? $1 + \frac{2}{n}$

   What is the height of the first right-hand rectangle? $\left(1+\frac{2}{n}\right)^2 + 2\left(1+\frac{2}{n}\right)$

   (c) What is the right-hand endpoint of the second subinterval? $1 + \frac{2}{n} + \frac{2}{n} = 1 + 2\left(\frac{2}{n}\right)$

   What is the height of the second right-hand rectangle? $f\left(1 + 2\left(\frac{2}{n}\right)\right)$

   (d) What is the right-hand endpoint of the third subinterval? $1 + \frac{2}{n} + \frac{2}{n} + \frac{2}{n} = 1 + 3\left(\frac{2}{n}\right)$

   What is the height of the third right-hand rectangle? $f\left(1 + 3\left(\frac{2}{n}\right)\right)$

   (e) What is the right-hand endpoint of the i^{th} rectangle? $1 + i\left(\frac{2}{n}\right)$

   What is the height of the i-th right-hand rectangle? $f\left(1 + i\left(\frac{2}{n}\right)\right)$

   What is the **area** of the i-th right-hand rectangle? $\frac{2}{n}\left(f\left(1 + i\left(\frac{2}{n}\right)\right)\right)$

Using #2,

3. Write down a limit that equals $\int_1^{\cancel{4}\,3} x^2 - 2x\,dx$. $= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2}{n}\left(\left(1+i\left(\frac{2}{n}\right)\right)^2 + 2\left(1+i\left(\frac{2}{n}\right)\right)\right)\right).$

4. Write down a limit that equals $\int_{\cancel{0}\,2}^{\cancel{5}\,8} e^x\,dx$, using right-hand endpoints as your sample points.

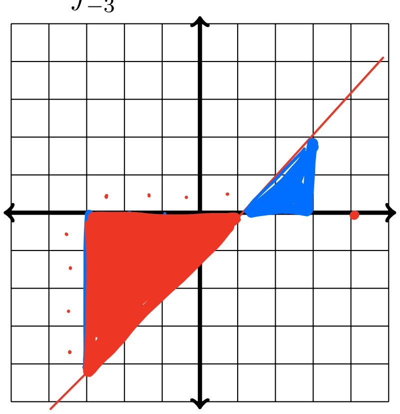
$\Delta x = \frac{8-2}{n} = \frac{5}{n}$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{5}{n}\left(e^{2+i(5/n)}\right)\right)$

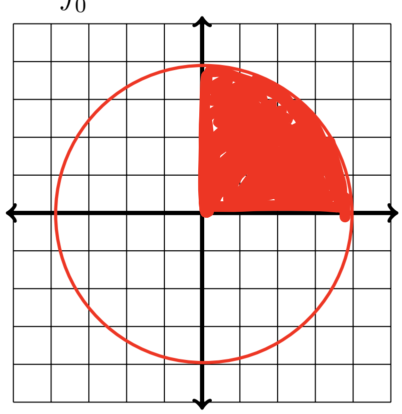
> *A definite integral represents the* **signed** *area under a curve (that is, the signed area between the curve and the x-axis). If a curve is above the x-axis that area is* positive*; if the curve is below the x-axis the area is* negative.

5. Evaluate the following definite integrals by drawing the function and interpreting the integral in terms of areas. Shade in the area you are computing with the integral.

(a) $\int_{-3}^{\cancel{1}\,3} (x-1)\,dx =$ $-8 + 2 = -6$



(b) $\int_0^4 \sqrt{16-x^2}\,dx =$ 4π

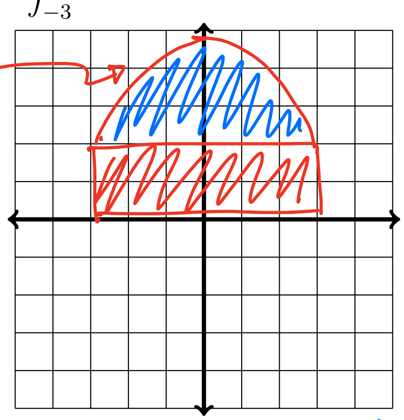


area $= \frac{1}{4}\pi(4)^2$

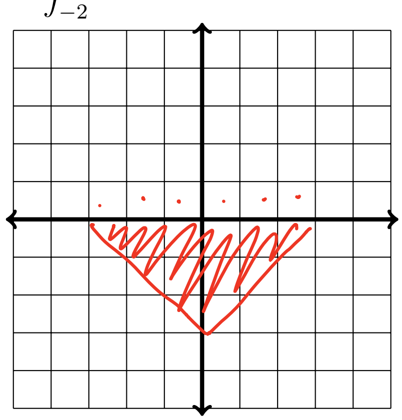
$= 4\pi$

(c) $\int_{-3}^{3} (2+\sqrt{9-x^2})\,dx =$ $12 + \frac{9}{2}\pi$

a semi-circle



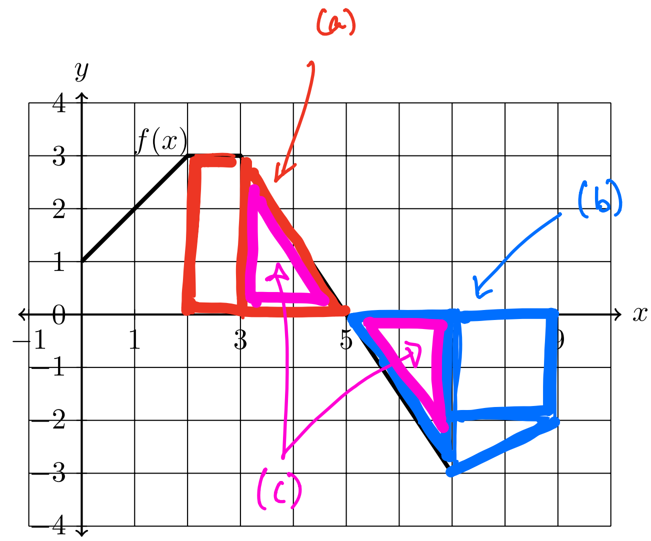
$\int_{-2}^{3} (|x|-3)\,dx =$ $\frac{1}{2}(6)(3) = 9$



Semicircle area $= \frac{\pi(9)}{2} = 9/2\,\pi$

 rectangle area $= 12$

6. The graph of f is shown. Evaluate each integral by interpreting it in terms of areas.



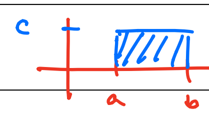
(a) $\int_2^5 f(x)dx =$ $3 + \frac{1}{2}(3)(2) = 6$

(b) $\int_5^9 f(x)dx =$ $\frac{1}{2}(2)(3) + 2(2)$
$+ \frac{1}{2}(2)(1) = 3 + 4 + 1 = 8$

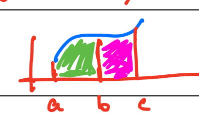
(c) $\int_3^7 f(x)dx =$ 0

Properties of the Definite Integral:

- $\int_a^b f(x)\,dx =$ area between the curve & x-axis on interval $[a,b]$
- $\int_a^a f(x)\,dx =$ 0 (no area is accumulated)
- $\int_a^b c\,dx =$ $c(b-a)$



- $\int_a^b cf(x)\,dx =$ $c\int_a^b f(x)\,dx$
- $\int_a^b [f(x) \pm g(x)]dx =$ $\left(\int_a^b f(x)\,dx\right) \pm \left(\int_a^b g(x)\,dx\right)$
- $\int_a^b f(x) + \int_b^c f(x)\,dx =$ $\int_a^c f(x)\,dx$



- $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

7. Using the fact that $\int_0^1 x^2 dx = \frac{1}{3}$ and $\int_1^2 x^2\,dx = \frac{7}{3}$, evaluate the following using the properties of integrals.

(a) $\int_1^0 x^2\,dx$

$= -\frac{1}{3}$

(b) $\int_0^1 5x^2\,dx$

$= 5\int_0^1 x^2\,dx$

$= 5\left(\frac{1}{3}\right) = \frac{5}{3}$

(c) $\int_0^1 (4 + 3x^2)\,dx$

$= \int_0^1 4\,dx + 3\int_0^1 x^2\,dx$

$= 4(1-0) + 3\left(\frac{1}{3}\right)$

$= 5$

(d) $\int_0^2 x^2\,dx.$

$= \int_0^1 x^2\,dx + \int_1^2 x^2\,dx$

$= \frac{1}{3} + \frac{7}{3}$

$= \frac{8}{3}$