Section 4.8 Newton's Method

1. Newton's Method is an iterative rule for finding roots.

Given: $F(x)$
Want: a so that $F(a) = 0$
Guess: x_0 close to a

Plug in and Repeat: $$x_{k+1} = x_k - \frac{F(x_k)}{F'(x_k)}$$

2. Let $F(x) = x^2 - 2$.

   (a) Using elementary algebra, find a such that $F(a) = 0$. (Find a exactly and find a decimal approximation with at least 9 decimal places.)

   $0 = a^2 - 2$ or $a = \pm 1.414213562$

   $a = \pm\sqrt{2}$

   SIMPLIFY

   (b) Find a formula for x_{k+1} :

   $F'(x) = 2x$

   $$x_{k+1} = x_k - \frac{(x_k^2 - 2)}{2x_k} = \underbrace{\frac{x_k}{2} + \frac{1}{x_k}}_{}$$

   !!!

   (c) Using an initial guess of $x_0 = 2$, complete 4 iterations of Newton's method to find x_4 and compare your answer to the one in part (a).

   $x_0 = 2$

   $x_1 = \frac{x_0}{2} + \frac{1}{x_0} = \frac{2}{2} + \frac{1}{2} = 1.5$

   color-coded to page 2

   $x_2 = \frac{x_1}{2} + \frac{1}{x_1} = \frac{1.5}{2} + \frac{1}{1.5} = 1.41\overline{6}$

   $x_3 = \frac{x_2}{2} + \frac{1}{x_2} = \frac{1.41\overline{6}}{2} + \frac{1}{1.41\overline{6}} = 1.414215686$

   Whoa.

   $x_4 = \frac{x_3}{2} + \frac{1}{x_3} = 1.414213562$

3. This page is intended to illustrate *how* Newton's Method works and *why* it has the formula it does.

Again, consider the function $F(x) = x^2 - 2$.

(a) Find the linearization $L(x)$ of $F(x)$ at $x = 2$. Leave your answer in point-slope form.

$F(2) = 2^2 - 2 = 4 - 2 = 2 \leftarrow$ y-value

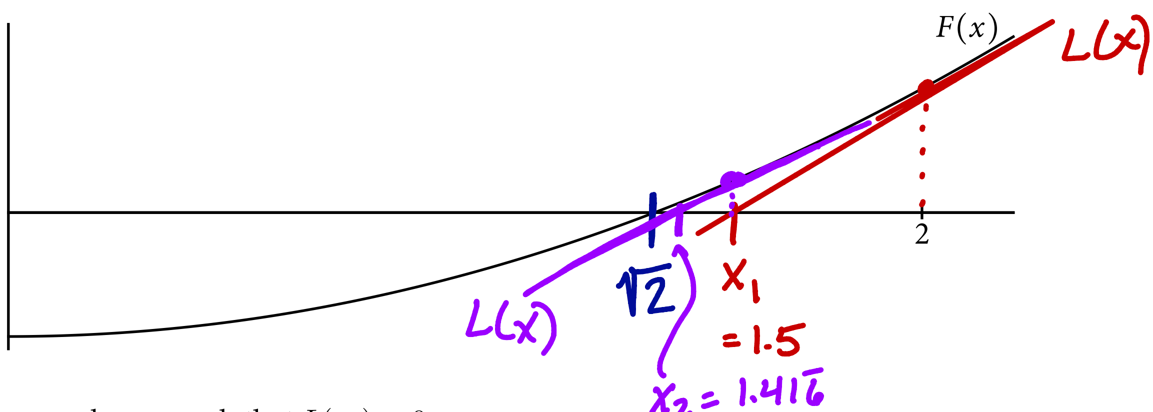
$F'(x) = 2x$

$F'(2) = 4 \leftarrow$ slope

$y - 2 = 4(x-2)$

$\boxed{L(x) = 2 + 4(x-2)}$

(b) I've graphed $F(x)$ for you below. Mark where $\sqrt{2}$ is on this diagram and add to this diagram the graph of $L(x)$.



(c) Find the number x_1 such that $L(x_1) = 0$.

$0 = 2 + 4(x_1 - 2)$

$-\frac{2}{4} = x_1 - 2$

So $x_1 = 2 - \frac{1}{2} = \frac{3}{2} = \boxed{1.5}$

(d) In the diagram above, label the point x_1 on the x-axis. ✓

(e) Let's do it again! Find the linearization $L(x)$ of $F(x)$ at $x = x_1$.

$x_1 = \frac{3}{2}$

$F\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - 2 = \frac{9}{4} - 2 = \frac{1}{4} \leftarrow$ y-value

$F'\left(\frac{3}{2}\right) = 3 \leftarrow$ slope

$y - \frac{1}{4} = 3\left(x - \frac{3}{2}\right)$

$\boxed{L(x) = \frac{1}{4} + 3\left(x - \frac{3}{2}\right)}$

(f) Add the graph of this new linearization to your diagram on the first page. ✓

(g) Find the number x_2 such that $L(x_2) = 0$. Then label the point $x = x_2$ in the diagram.

$0 = \frac{1}{4} + 3\left(x_2 - \frac{3}{2}\right)$

$-\frac{1}{12} = x_2 - \frac{3}{2}$

So $x_2 = \frac{3}{2} - \frac{1}{12} = \frac{17}{12} = 1.41\overline{6}$

(h) Compare your numbers for x_1 and x_2 to those on the previous page. They should be the same.

Yep.

(i) Let's be a little more systematic. Suppose we have an estimate x_k for $\sqrt{2}$.

- Compute $F(x_k)$. $x_k^2 - 2$ ← y-value
- Compute $F'(x_k)$. $2x_k$ ← slope
- Compute the linearization of $F(x)$ at $x = x_k$.

$$y - (x_k^2 - 2) = 2x_k(x - x_k)$$

$$L(x) = (x_k^2 - 2) + 2x_k(x - x_k)$$

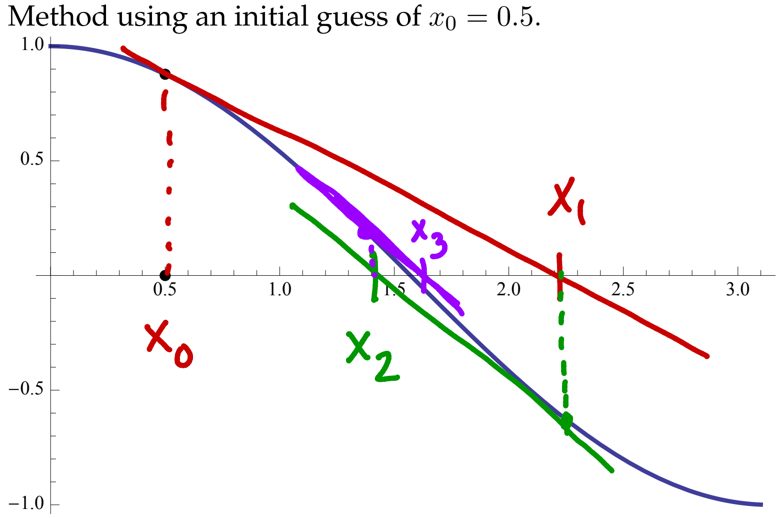
- Find the number x_{k+1} such that $L(x_{k+1}) = 0$. You should try to find as simple an expression as you can. Compare this to the formula we used on problem 1b from page 1.

$$0 = (x_k^2 - 2) + 2x_k(x_{k+1} - x_k)$$

$$-\frac{x_k^2 - 2}{2x_k} = x_{k+1} - x_k$$

$$x_{k+1} = x_k - \frac{x_k^2 - 2}{2x_k} = \frac{x_k}{2} + \frac{1}{x_k}$$ ← !!!

4. Indicate on the picture below, the values of x_1, x_2 and x_3 that would be obtained from Newton's Method using an initial guess of $x_0 = 0.5$.



5. Try to solve

$$e^{-x} - x = 0$$

by hand.

Try $e^{-x} = x$

$-x = \ln x$ ... ugh...

6. Explain why there is a solution between $x = 0$ and $x = 1$.

$x=0$: $e^{-x} - x = e^{0} - 0 = 1 > 0$

$x=1$: $e^{-x} - x = e^{-1} - 1 = \frac{1}{e} - 1 < 0$

So $F(x) = e^{-x} - x$ must cross x-axis between $x=0$ and $x=1$

7. Starting with $x_0 = 1$, find an approximation of the solution of $e^{-x} - x = 0$ to 6 decimal places. During your computation, keep track of each x_k to at least 9 decimal places of accuracy.

$$x_{k+1} = x_k - \frac{e^{-x_k} - x_k}{-e^{-x_k} - 1} = x_k + \frac{e^{-x_k} - x_k}{e^{-x_k} + 1} \cdot \frac{e^{+x_k}}{e^{+x_k}}$$

$$= x_k + \frac{1 - x_k e^{x_k}}{1 + e^{x_k}}$$

$$x_1 = x_0 + \frac{1 - x_0 e^{x_0}}{1 + e^{x_0}} = 1 + \frac{1-e}{1+e} \approx 0.537882843\ldots$$

$x_2 \approx 0.566986991\ldots$

$x_3 \approx 0.567143268\ldots$

$x_4 \approx 0.567143290\ldots$

quick check: $e^{-0.5671432} - 0.5671432 = 1.4 \times 10^{-7}$:)