SECTION 2-6 LIMITS AT INFINITY (DAY 2): SOMETIMES WE HAVE TO USE TRICKS!

1. Multiply the top and bottom by $\frac{1}{e^x}$, to help compute:

$$\lim_{x\to\infty}\frac{1+5e^x}{7-e^x} = \lim_{x\to\infty}\frac{1/e^x+5}{7/e^x-1} = \frac{\left(\lim_{x\to\infty}\overset{\nearrow 0}{1/e^x}\right)+5}{\left(\lim_{x\to\infty}\underset{\searrow 0}{7/e^x}\right)-1} = -5$$

2. This time we need two tricks: (1) rewrite the difference of natural logs as a quotient, and (2) use the continuity of natural log to pull it through the limit, to help compute:

$$\lim_{x\to\infty}[\ln(2+3x)-\ln(1+x)] = \lim_{x\to\infty}\ln\left(\frac{2+3x}{1+x}\right)$$

$$= \ln\left(\lim_{x\to\infty}\frac{2+3x}{1+x}\right) = \ln\left(\lim_{x\to\infty}\frac{2/x+3}{1/x+1}\right) = \ln(3)$$

3. Even though the x^6 is part of a term in the square root, as x blows up, $\sqrt{x^6+\ldots}$ "looks like" x^3. So try multiplying the top and bottom by $\frac{1}{x^3}$ (which in this case is the same as $\sqrt{\frac{1}{x^6}}$).

$$\lim_{x\to\infty}\left(\frac{\sqrt{3x^6-x}}{x^3+1}\right)\left(\frac{1/x^3}{1/x^3}\right) = \lim_{x\to\infty}\frac{\sqrt{3x^6/x^6-\frac{x}{x^6}}}{x^3/x^3+1/x^3} = \lim_{x\to\infty}\frac{\sqrt{3-\frac{1}{x^5}}}{1+1/x^3}$$

$$= \frac{\sqrt{3-\lim_{x\to\infty}1/x^5}}{1+\lim_{x\to\infty}1/x^3} = \frac{\sqrt{3-0}}{1+0} = \sqrt{3}$$

Observe that the problem below looks just like the one above but with one small difference. Look and think carefully before you evaluate.

$$\lim_{x\to-\infty}\frac{\sqrt{3x^6-x}}{x^3+1} = \lim_{x\to\infty}\frac{\sqrt{3(-x)^6-(-x)}}{(-x)^3+1} = \lim_{x\to\infty}\left(\frac{\sqrt{3x^6+x}}{-(x^3)+1}\right)\frac{1/x^3}{1/x^3}$$

$$= \lim_{x\to\infty}\frac{\sqrt{3+1/x^5}}{-1+1/x^3} = -\sqrt{3}$$

4. If, when you think about a limit, it "looks like" $\infty - \infty$, that means you need to do more work. Hint: The value of this limit is *not* zero.

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)\left(\frac{\sqrt{x^2+x}\quad +x}{\sqrt{x^2+x}\quad +x}\right)$$

$$= \lim_{x\to\infty}\left(\frac{\cancel{x^2}+x-\cancel{x^2}}{\sqrt{x^2+x}\;+x}\right)\frac{1/x}{1/x} = \lim_{x\to\infty}\frac{1}{\sqrt{1+1/x}\;+1} = \frac{1}{\sqrt{1+0}\;+1} = \frac{1}{2}$$

5. We know that $-1 \le \cos(x) \le 1$. Use that fact plus the *Squeeze Theorem* to evaluate the following:

$\lim_{x\to\infty} e^{-2x}\cos x$ Note $-1 \le \cos(x) \le 1$, so $-\frac{1}{e^{2x}} \le e^{-2x}\cos(x) \le \frac{1}{e^{2x}}$

And therefore $\lim_{x\to\infty} \frac{-1}{e^{2x}} \le \lim_{x\to\infty} e^{-2x}\cos(x) \le \lim_{x\to\infty}\frac{1}{e^{2x}}$. $\Rightarrow$

$0 \le \lim_{x\to\infty} e^{-2x}\cos(x) \le 0$. By the squeeze theorem,

$\lim_{x\to\infty} e^{-2x}\cos(x) = 0$.

6. Find any horizontal or vertical asymptotes of the curve below. If none exists, state that explicitly. On quizzes and tests, you will be asked to show your work, so practice that now!

$$f(x) = y = \frac{2x^2-x-1}{3x^2-2x-1} = \frac{(2x+1)(x-1)}{(3x+1)(x-1)}$$

$\leftarrow$ undefined at $x=1$ and $x=-1/3$

$\lim_{x\to 1} f(x) = \lim_{x\to 1}\frac{2x+1}{3x+1} = \frac{2}{3}$ so there's a $\boxed{\text{hole at } x=1}$

$-1/4$

$-1/3 \quad 0$

$\lim_{x\to -1/3^+} f(x) = \lim_{x\to -1/3^+}\frac{2x^2-x-1}{3x^2-2x-1}$ $\rightarrow$ $2/9 + 1/3 - 1$ is finite & negative; $\rightarrow$ $1/3 + 2/3 - 1 = 0^+$ $= +\infty$ $\boxed{x=-1/3 \text{ is VA}}$

Check: $f(-.3) = \frac{.09-.3-1}{.3-.6-1}$ <0 <0

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{2x^2-x-1}{3x^2-2x-1} = \lim_{x\to\infty}\frac{2-1/x-1/x^2}{3-2/x-1/x^2} = 2/3$

$\lim_{x\to -\infty} f(x) = \lim_{x\to\infty}\frac{2(-x)^2-(-x)-1}{3(-x)^2-2(-x)-1} = 2/3$.

$\boxed{y = 2/3 \text{ is HA}}$

7. ~~In a differential equations course, you can prove that the velocity of a falling raindrop at time t is:~~

$$v(t) = k(1-e^{-gt/k})$$

~~where k is the terminal velocity of the raindrop and g is the acceleration due to gravity. (That is, k and g are positive fixed constants.)~~

(a) ~~Find $\lim_{t\to\infty} v(t)$~~

(b) ~~Interpret your answer above in the context of the problem.~~

(Needed more space.)