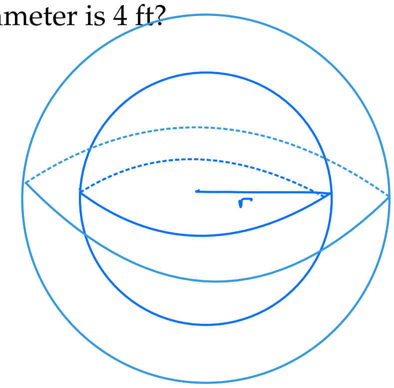
Here are the steps I will take to solve a problem like this:

1. (Draw a picture and label useful quantities.)
2. Identify what you $\boxed{\text{know.}}$
3. Identify the quantity you $\boxed{\text{want.}}$
4. Find an $\boxed{\text{equation that relates the knowns and the wants.}}$
5. Now differentiate $\boxed{\text{implicitly with respect to } t.}$
6. Plug in and solve.

We'll repeat this procedure with a bunch of examples.

1. Air is being pumped into a spherical balloon so that its volume increases at a rate of 4.5 ft^3/min. How fast is the radius of the balloon increasing when the diameter is 4 ft?



Know $\boxed{\frac{dV}{dt} = 4.5\ \text{ft}^3/\text{min}}$. Want $\frac{dr}{dt}$ when

diam = 4 ft $\Rightarrow$ $\boxed{r = 2\text{ ft}}$

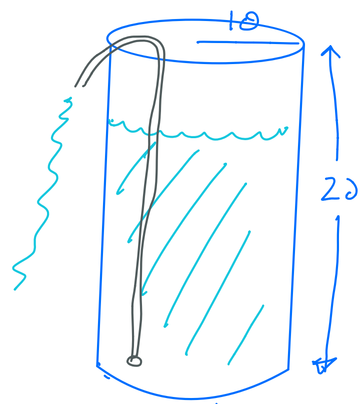
Also, $V = \frac{4}{3}\pi r^3$

So $\boxed{\frac{dV}{dt}} = \frac{4}{3}\pi \cdot \boxed{3r^2} \cdot \frac{dr}{dt}$.

Therefore,

$4.5 = \frac{4}{\cancel{3}}\pi(\cancel{3}(2)^2)\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{4.5}{16\pi}$

2. Water is being pumped out of a cylindrical tank with a radius of 10 meters and a height of 20 meters. The height of the water is decreasing at a rate of 0.2 meters per minute. At what rate is the water being pumped out?



Know $\frac{dh}{dt} = 0.2$ m/min. Want $\frac{dV}{dt}$

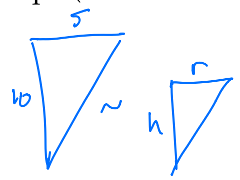
Know $V = \pi r^2 h$ but $r = 10$ so $V = 100\pi h$

$\frac{dV}{dt} = 100\pi\frac{dh}{dt}$

So $\frac{dV}{dt} = 100\pi(0.2) = 20\pi\ \text{m}^3/\text{min}$

$100 \cdot \frac{2}{10} = 20$

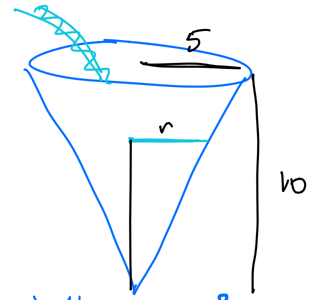
3. Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep? (Hint: Use similar triangles.)



Know $h = 6$

$\frac{dV}{dt} = 9 \text{ ft}^3/\text{min}$

Want $\boxed{\frac{dh}{dt}}$



But $V = \frac{1}{3}\pi r^2 h$ and $\frac{r}{h} = \frac{5}{10} \Rightarrow r = \frac{h}{2}$

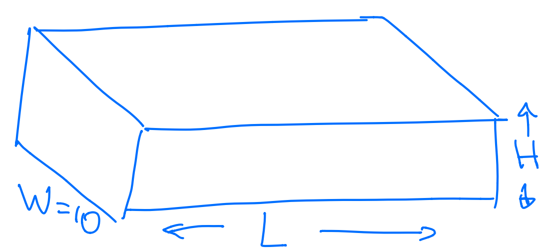
So $V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 \cdot h = \frac{1}{3} \cdot \pi \frac{h^3}{4} = \frac{\pi h^3}{12} \Rightarrow \frac{dV}{dt} = \frac{\pi}{12}(3h^2)\frac{dh}{dt} = \frac{\pi h^2}{4} dh/dt$

So when $\frac{dV}{dt} = 9$, $9 = \frac{\pi(6)^2}{4}\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{9 \cdot 4}{36\pi} = \frac{1}{\pi}$

4. A rectangular solid has a fixed width of 10 cm but its length is increasing at a rate of 2 cm/s while is height is decreasing at a rate of 1 cm/s. At what rate is the volume of the solid changing when the length is 20 cm and the height is 15 cm?

Know $\frac{dL}{dt} = 2 \text{ cm/s}$, $\frac{dH}{dt} = -1 \text{ cm/s}$,

$W = 10$



Know: $W \cdot L \cdot H = V \Rightarrow V = 10LH$

Want $\frac{dV}{dt}$. But $\frac{dV}{dt} = 10\left(L\frac{dH}{dt} + H\frac{dL}{dt}\right)$, so

when $L = 20$ & $H = 15$, $\frac{dV}{dt} = 10\left(20(-1) + 15(2)\right) = 10(-20+30) = 100 \text{ cm}^3/\text{s}$

5. The standard 12 foot ladder rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

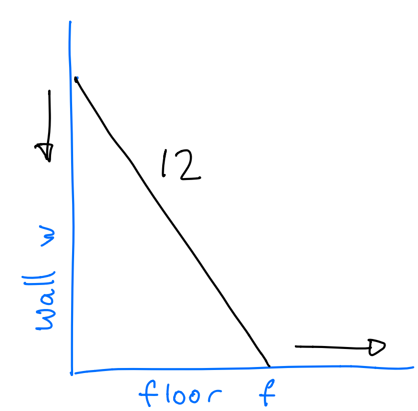
Know $\frac{df}{dt} = 1 \text{ ft/s}$. Want $\frac{dw}{dt}$ when $f = 6$. Note when $f = 6$, $w = \sqrt{144-36} = 6\sqrt{3}$

Know $f^2 + w^2 = 144$. So

$2f\frac{df}{dt} + 2w\frac{dw}{dt} = 0 \Rightarrow$

$2(6)(1) + 2(6\sqrt{3})\frac{dw}{dt} = 0 \Rightarrow$

$\frac{dw}{dt} = \frac{-2 \cdot 6}{2 \cdot 6\sqrt{3}} = \frac{-1}{\sqrt{3}} \approx -0.577 \text{ ft/s}$



That is, the top is sliding down at a rate of $1/\sqrt{3}$ ft/s

6. A hot air balloon rising straight up from a level field is tracked by a range finder 500 feet from the lift-off point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 radians/min. How fast is the balloon rising at that moment?

Know $\frac{d\theta}{dt} = 0.14$ rad/min when $\theta = \pi/4$. Want $\frac{dh}{dt}$.

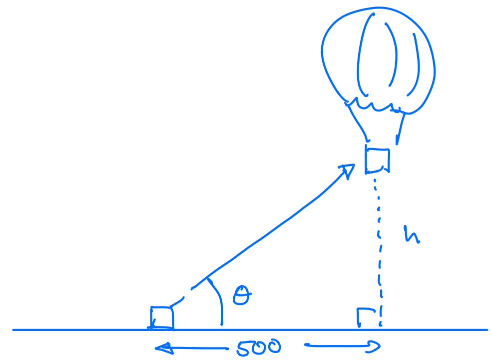
Relate: $\frac{h}{500} = \frac{opp}{adj} = \tan\theta \Rightarrow h = 500\tan\theta$

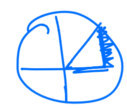
So $\frac{dh}{dt} = 500(\sec\theta)^2 \frac{d\theta}{dt} \Rightarrow$

$\frac{dh}{dt} = 500(\sec(\pi/4))^2(0.14)$

$= 500\left(\frac{1}{1/\sqrt{2}}\right)^2(0.14) = 500(2)(0.14)$

$= 140$ ft/min





7. A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine that the distance between them and the car they are chasing is increasing at a rate of 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car? [Hint: You'll need to relate *three* quantities here!]

Know $\boxed{\frac{dP}{dt} = -60 \text{ miles/hour}}$ Want $\frac{ds}{dt}$ → Why negative? Because the police cruiser is going south.

$\frac{dD}{dt} = 20$ miles/hour when $p = 0.6$ and $s = 0.8$

Know $s^2 + p^2 = D^2$. So when $p = \frac{6}{10} = \frac{3}{5}$ & $s = \frac{8}{10} = \frac{4}{5}$

$D = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$, and

$2s\frac{ds}{dt} + 2p\frac{dP}{dt} = 2D\frac{dD}{dt} \Rightarrow$

$2\left(\frac{4}{5}\right)\frac{ds}{dt} + 2\left(\frac{3}{5}\right)(-60) = 2(1)(20) \Rightarrow$

$\frac{8}{5}\frac{ds}{dt} = 40 + 72 = 112$

So $\frac{ds}{dt} = \frac{112 \cdot 5}{8} = 70$ miles/hour.