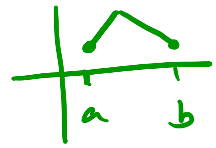
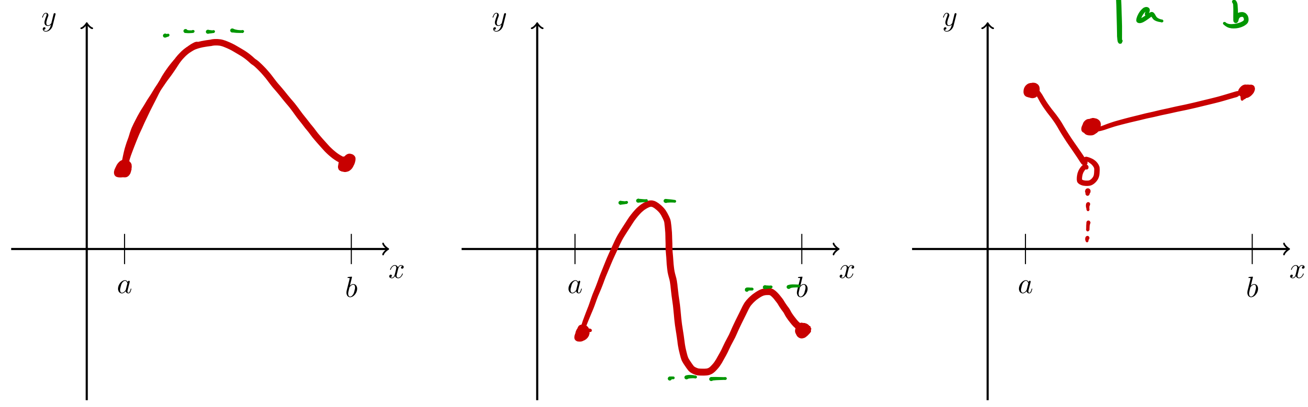
4-2 THE MEAN VALUE THEOREM

MOTIVATING EXAMPLES: Draw several examples of graphs of functions such that (i) the domain is $[a, b]$ and (ii) $f(a) = f(b)$. Note you are not *required* to make sketches that are continuous or differentiable, though you may choose to do so.





Draw some conclusions:

If $f(x)$ is continuous + differentiable, then there is some place where the tangent is horizontal

ROLLE'S THEOREM: If

① $f(x)$ is continuous on $[a,b]$ and
② $f'(x)$ exists on (a,b) and
③ $f(a)=f(b)$

then there is a number c in the interval (a, b) such that $f'(c) = 0$.

PROOF: What can happen as you trace the graph from $x=a$ to $x=b$?

- It stays flat. So every c in (a,b) has $f'(c)=0$.
- It starts to increase. So f has a max.
- It starts to decrease. So f has a min.

So at the x-values (say $x=c$) where these occur, $f'(c)=0$.

PRACTICE PROBLEMS:

1. Consider $f(x) = x^4 - \frac{8}{3}x^3 + 1$ on the interval $[0, 8/3]$.

   (a) Verify that the function $f(x)$ satisfies the hypothesis of Rolle's Theorem on the given interval.

③ $f(0) = 1$

$f(8/3) = 1$

① and ② : f is a polynomial so it is continuous and differentiable everywhere.

   (b) Find all numbers c that satisfy the conclusion of Rolle's Theorem.

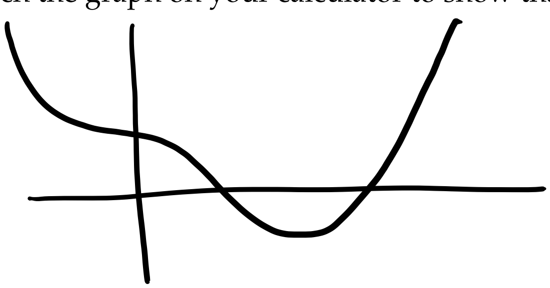
$f'(x) = 4x^3 - 8x^2 = 4x^2(x-2)$

$f'(x) = 0$ when $x = 0$ or $x = 2$

not in interval $(0, 8/3)$

Answer : $c = 2$

   (c) Sketch the graph on your calculator to show that your answer above are correct.

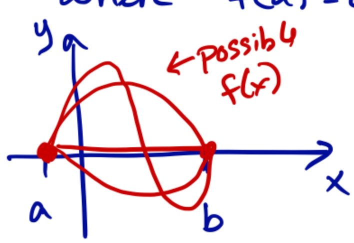


3.5π

$= 2\pi + 1.5\pi$

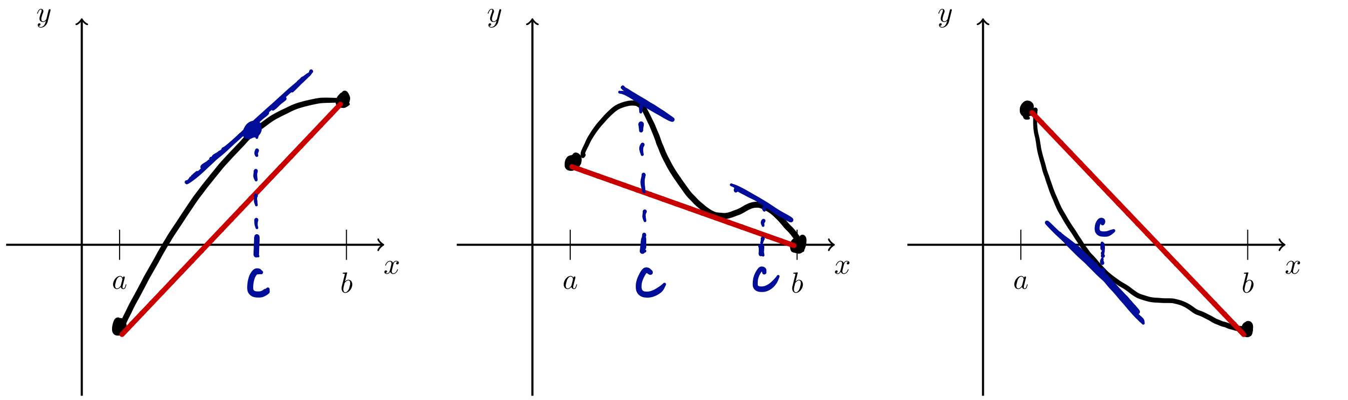
2. *Use Rolle's Theorem* to show that the equation $x^3 - 15x + d = 0$ can have at most one solution in the interval $[-2, 2]$.
   HINT: Show that there is no way there could be two solutions!

OK. I'll follow the hint. What if $f(x) = x^3 - 15x + d$ has two solutions in $[-2,2]$? Then $f(x)$ would have two x-values (say $x=a$ + $x=b$) where $f(a) = 0 = f(b)$. [Rolle's Thm uses $\boxed{f'}$ so I'll find that.]



Now $f'(x) = 3x^2 + 15 = 0$ if $x = \pm\sqrt{5}$. But neither $\sqrt{5}$ or $-\sqrt{5}$ are in $[-2,2]$. So $f(x)$ has no turn around points. So it can't have two solutions.

MOTIVATING EXAMPLES: Draw several examples of graphs of functions such that (i) the domain is $[a, b]$, (ii) $f(x)$ is continuous on $[a, b]$, **and** (iii) $f(x)$ is differentiable on $[a, b]$. We are *not* assuming that $f(a) = f(b)$.



QUESTION: In each picture above, draw (or in some other way identify) the quantity:

$$\frac{f(b) - f(a)}{b - a}.$$

* It's the slope of the lines.

What would this quantity be if Rolle's Theorem applied?

THE MEAN VALUE THEOREM: If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there is a number c in the interval (a, b) such that

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}.$$

QUESTION: Assume that $f(x)$ is continuous and differentiable on the interval $[a, b]$ and assume there exists some x-value d in (a, b) such that $f(d) > f(a)$, can you draw any conclusion about $f'(x)$? What if $f(d) < f(a)$?

If $f(d) > f(a)$, there has to be someplace where $f'(x) > 0$ on $[a,b]$.

If $f(d) < f(a)$, there has to be someplace where $f'(x) < 0$ on $[a,b]$.

THEOREM 5: If $f'(x) = 0$ for all x in the interval (a, b), then

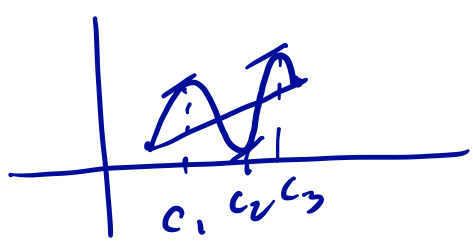
$f(x)$ is constant or
$f(x) = C$ some real number c

.

QUESTION: If $f(x)$ gives the *position* of an object as a function of time, what "common sense" idea is the MVT telling us? Theorem 5?

On a trip, at some point, your car is going your average speed.
If your velocity is always 0, you didn't go anywhere.

1. Sketch the graph of a function $f(x)$ on an interval $[0, 5]$ such that there are exactly three numbers c in $(0, 5)$ satisfying the Mean Value Theorem.



2. Suppose that $f(0) = -3$ and that $f'(x)$ exists and is less than or equal to 5 for all values of x. How large can $f(2)$ possibly be?

$-3 + 2 \cdot 5 = 7$

3. Consider $f(x) = x^{-2}$ on the interval $[1, 2]$.

(a) Verify that the function $f(x)$ satisfies the hypothesis of the Mean Value Theorem on the given interval.

f is continuous + differentiable on $[1,2]$.

(b) Find all numbers c that satisfy the conclusion of the Mean Value Theorem.

$f(1) = 1$, $f(2) = \frac{1}{4}$

$$\frac{f(2) - f(1)}{2 - 1} = \frac{1}{4} - 1 = \frac{-3}{4}$$

$f' = -2x^{-3}$

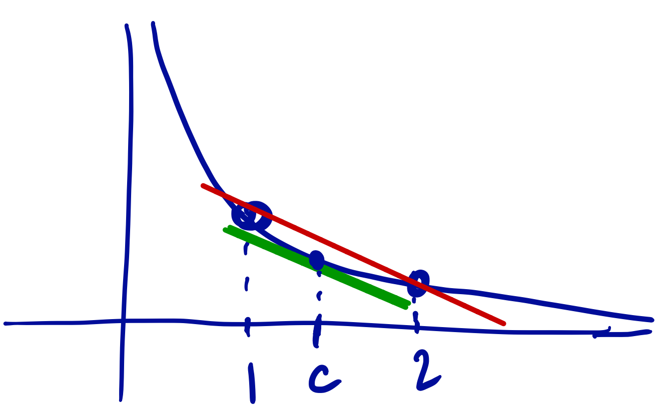
$$\frac{-2}{x^3} = \frac{-3}{4}$$

So $\frac{8}{3} = x^3$

or $x = \frac{2}{\sqrt[3]{3}}$

answer : $c = 2/\sqrt[3]{3}$

(c) Sketch the graph to show that your answer above are correct.



ONE LAST BIG IDEA:

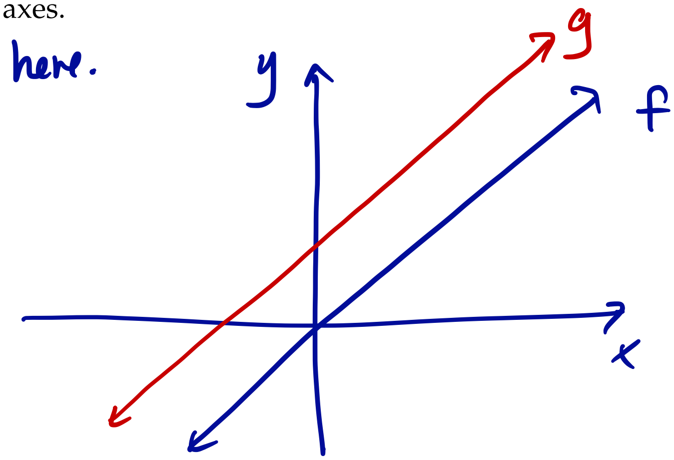
1. Give the formulas for two *different* functions $f(x)$ and $g(x)$ such that $f'(x) = g'(x)$ and sketch these two functions on the same set of axes.

Many correct answers here.

$f(x) = x$

$g(x) = x+1$

$f'(x) = 1 = g'(x)$



2. **Corollary 7:** If $f'(x) = g'(x)$ for all x in the interval (a, b), then

$$\boxed{f(x) = g(x) + C, \quad C \text{ some constant}}$$

or, said another way,

$$\boxed{f(x) - g(x) = C}$$

i.e. f & g are a fixed vertical distance away from each other.

3. Why is Corollary 7 true?

If $f' = g'$, then $f' - g' = \boxed{0}$? (algebra)

If $H'(x) = 0$ everywhere then $H(x) = \boxed{C}$? (Thm 5)

a constant.

What is $\frac{d}{dx}[f(x) - g(x)] = \boxed{f'(x) - g'(x)}$?

Difference Rule