4-4: L'HOSPITAL'S RULE

1. Compare the following two limits:

factor

(a) $\lim_{x\to 4}\frac{2x^2-5x-12}{x^2-3x-4} = \cancel{\frac{32-20-12}{16-12-4} = \frac{0}{0}} = \lim_{x\to 4}\frac{(2x+3)(x-4)}{(x-4)(x+1)} = \lim_{x\to 4}\frac{2x+3}{x+1} = \frac{11}{5}$

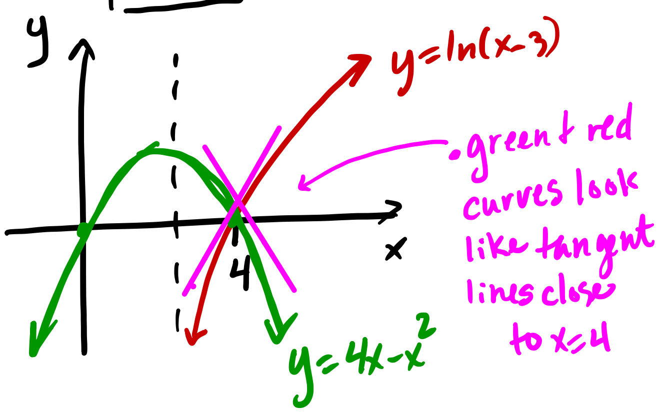
plug in   bad   why?   plug in

(b) $\lim_{x\to 4}\frac{\ln(x-3)}{4x-x^2} = \cancel{\frac{\ln 1}{16-16} = \frac{0}{0}}$ // $\lim_{x\to 4}\frac{\ln(x-3)}{4x-x^2} \overset{H}{=} \lim_{x\to 4}\frac{\frac{1}{x-3}}{4-2x}$

plug in   bad.

$= \lim_{x\to 4}\frac{\frac{1}{1}}{4-8} = -\frac{1}{4}$

picture:



2. L'Hospital's Rule

• If $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

provided the limit on the right exists. (or is $\pm\infty$)

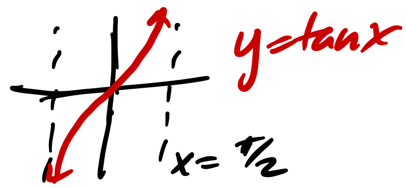
• If $\lim_{x\to a} f(x)=\pm\infty$ and $\lim_{x\to a} g(x)=\pm\infty$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

provided the limit on the right exists. (or is $\pm\infty$)

3. (Some routine examples.) Evaluate the limits.

(a) $\lim\limits_{x\to(\pi/2)^+} \dfrac{\cos x}{1-\sin x} \overset{H}{=} \lim\limits_{x\to\frac{\pi}{2}^+} \dfrac{-\sin x}{-\cos x} = \lim\limits_{x\to\frac{\pi}{2}^+} \tan x = +\infty$

form $\frac{0}{0}$



(b) $\lim\limits_{x\to\infty} \dfrac{\ln\sqrt{x}}{x^2} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{2}\ln x}{x^2} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{2}\cdot\frac{1}{x}}{2x} = \lim\limits_{x\to\infty} \dfrac{1}{4x^2} = 0$

form $\frac{\infty}{\infty}$

(c) $\lim\limits_{x\to 5^+} \dfrac{e^x-1}{x-5} = +\infty$

form $\frac{e^5-1}{0}$. L'Hospital's does not apply!

as $x\to 5^+$, $x-5\to 0^+$; as $x\to 5^+$, $e^x-1\to e^5-1$.

(d) $\lim\limits_{x\to\infty} \dfrac{e^x}{x^2} \overset{H}{=} \lim\limits_{x\to\infty} \dfrac{e^x}{2x} \overset{H}{=} \lim\limits_{x\to\infty} \dfrac{e^x}{2} = \frac{1}{2}\lim\limits_{x\to\infty} e^x = \infty.$

form $\frac{\infty}{\infty}$ form $\frac{\infty}{\infty}$

4. L'Hospital's Rule can address other indeterminate forms.

like $0\cdot\infty$, $\infty-\infty$, 1^∞, 0^0, ∞^0

5. Examples to demonstrate.

(a) $\lim\limits_{x\to 0^+} x\ln x = \lim\limits_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x}} = \lim\limits_{x\to 0^+} \dfrac{\ln x}{x^{-1}} \overset{H}{=} \lim\limits_{x\to 0^+} \dfrac{\frac{1}{x}}{-1x^{-2}} = \lim\limits_{x\to 0^+} -x = 0.$

form $0\cdot(-\infty)$

form $\frac{-\infty}{\infty}$

(b) $\lim\limits_{x\to\infty} \left(1+\dfrac{a}{x}\right)^{bx} = \boxed{e^{ab}}$

form 1^∞

If $y=(1+ax^{-1})^{bx}$, then $\ln y = bx\ln(1+ax^{-1})$.

Consider $\lim\limits_{x\to\infty} bx\ln(1+ax^{-1}) = \lim\limits_{x\to\infty} \dfrac{b\ln(1+ax^{-1})}{x^{-1}} \to$

6. Examples for you.

(a) $\lim\limits_{x\to 1^+} x^{\frac{1}{1-x}} = \boxed{e^{-1}}$

form 1^∞

$\lim\limits_{x\to 1^+} \left(\dfrac{1}{1-x}\right)\ln x = \lim\limits_{x\to 1^+} \dfrac{\ln x}{1-x} \overset{H}{=} \lim\limits_{x\to 1^+} \dfrac{\frac{1}{x}}{-1} = -1$

form $\frac{0}{0}$

(b) $\lim\limits_{x\to\infty} x^{\frac{3}{2}} \sin\left(\dfrac{1}{x}\right) = \lim\limits_{x\to\infty} \dfrac{\sin(x^{-1})}{x^{-3/2}} \overset{H}{=} \lim\limits_{x\to\infty} \dfrac{\cos(x^{-1})\cdot -x^{-2}}{-\frac{3}{2}x^{-5/2}}$

form $\infty\cdot 0$

form $\frac{0}{0}$

$= \lim\limits_{x\to\infty} \dfrac{2}{3} x^{\frac{1}{2}} \cos(1/x) = \infty.$

(c) $\lim\limits_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{e^x-1}\right) = \lim\limits_{x\to 0^+} \dfrac{e^x-1-x}{x(e^x-1)} \overset{H}{=} \lim\limits_{x\to 0^+} \dfrac{e^x-1}{1(e^x-1)+xe^x}$

$\infty - \infty$

form $\frac{0}{0}$

form $\frac{0}{0}$

$\overset{H}{=} \lim\limits_{x\to 0^+} \dfrac{e^x}{e^x+1\cdot e^x+xe^x} = \dfrac{1}{2}$

Consider $\lim\limits_{x\to\infty} bx\ln(1+ax^{-1}) = \lim\limits_{x\to\infty} \dfrac{b\ln(1+ax^{-1})}{x^{-1}}$

form $0/0$

$$\overset{H}{=} \lim_{x\to\infty} \frac{\frac{b}{1+ax^{-1}}\cdot -ax^{-2}}{-x^{-2}} = \lim_{x\to\infty} \frac{+ab}{1+ax^{-1}} = ab$$