Math F252 **Final Exam** **Fall 2025**

Name: Solutions

Rules:

You have 120 minutes to complete this midterm.

Partial credit will be awarded, but you must show your work.

Calculators are not allowed.

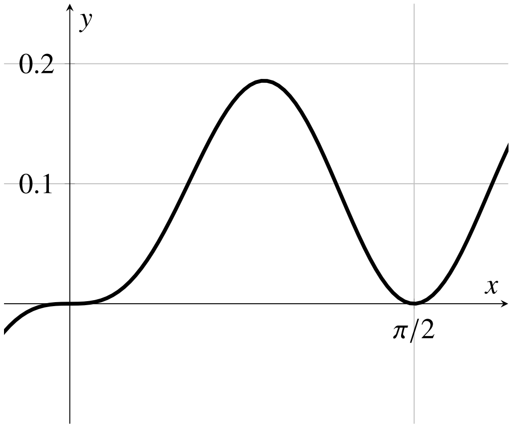
One hand-written sheet of notes is allowed.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	6	
2	6	
3	15	
4	10	
5	20	
6	6	
7	10	
8	10	
9	6	
10	5	
11	6	
Extra Credit	5	
Total	100	

1. (6 pts) The region R is bounded by $y = \sin^3(x)\cos^2(x)$ and the x-axis between $x = 0$ and $x = \pi/2$. Find the **area** of the region R.



$$A = \int_0^{\pi/2} \sin^3 x \cos^2 x \, dx = \int_0^{\frac{\pi}{2}} \sin^2 x \cos^2 x \sin x \, dx$$

let $u = \cos x$
$du = -\sin x \, dx$
if $x = 0$, $u = 1$.
if $x = \frac{\pi}{2}$, $u = 0$

$$= \int_0^{\pi/2} (1 - \cos^2 x)\cos^2 x \sin(x) \, dx$$

* I used this minus sign to reverse the order of integration

$$= \int_0^1 (1 - u^2) u^2 \, du$$

$$= \int_0^1 u^2 - u^4 \, du = \frac{1}{3}u^3 - \frac{1}{5}u^5 \Big|_0^1 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$$

2. (6 pts) The shaft of a bird feather has density function $\boxed{\rho = \frac{2}{3}\arctan x}$ grams per meter on the interval from $x = 0$ m to $x = 1$ m. Find the **mass** of the shaft. **Include units with your answer.**

$$\text{mass} = \int_0^1 \frac{2}{3}\arctan x\,dx = \frac{2}{3}\left[x\arctan x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx\right]$$

IBP

$u = \arctan x \qquad dv = dx$

$du = \frac{1}{1+x^2}\,dx \qquad v = x$

$$= \frac{2}{3}\left(\arctan(1)\right) - \frac{2}{3}\cdot\frac{1}{2}\cdot\ln(1+x^2)\Big|_0^1$$

$$= \frac{2}{3}\left(\frac{\pi}{4}\right) - \frac{1}{3}\left(\ln(2) - \ln(1)\right) = \boxed{\frac{\pi}{6} - \frac{1}{3}\ln(2)}$$ grams

3. (5 pts each) Evaluate the indefinite integrals.

(a) $\displaystyle\int \frac{1}{x^2\sqrt{x^2-9}}\,dx = \int \frac{3\sec\theta\tan\theta\,d\theta}{9\sec^2\theta\cdot 3\tan\theta} = \frac{1}{9}\int \frac{d\theta}{\sec\theta}$

trig substitution

let $x = 3\sec\theta$

$d\theta = 3\sec\theta\tan\theta\,d\theta$

$x^2 - 9 = 9\sec^2\theta - 9$

$= 9\tan^2\theta$

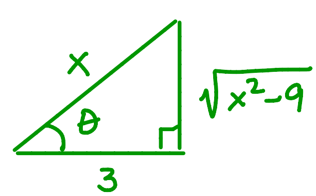
$\sqrt{x^2-9} = 3\tan\theta$

$$= \frac{1}{9}\int \cos\theta\,d\theta = \frac{1}{9}\sin\theta + C$$

$$= \boxed{\frac{1}{9}\left(\frac{\sqrt{x^2-9}}{x}\right) + C}$$

$\frac{x}{3} = \sec\theta$

$= \frac{\text{hyp}}{\text{adj}}$



(b) $\int \frac{1}{x^2-4}dx = -\frac{1}{4}\int \frac{dx}{x+2} + \frac{1}{4}\int \frac{dx}{x-2} = -\frac{1}{4}\ln|x+2| + \frac{1}{4}\ln|x-2| + C$

$$\boxed{= \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C}$$

partial fractions

$$\frac{1}{x^2-4} = \frac{A}{x+2} + \frac{B}{x-2}$$

$$1 = A(x-2) + B(x+2)$$

let $x=2$: $1 = 4B$, $B = 1/4$

let $x=-2$: $1 = -4A$; $A = -1/4$

(c) $\int (3x - xe^x)dx = \int 3x\,dx - \int xe^x dx$

IBP

$u = x$ $\quad dv = e^x dx$

$du = dx$ $\quad v = e^x$

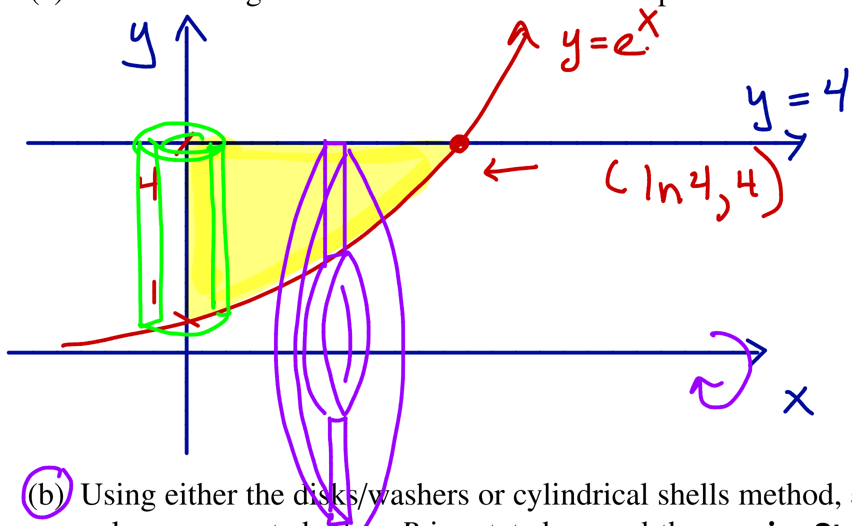
$$= \frac{3}{2}x^2 - \int xe^x dx$$

$$= \frac{3}{2}x^2 - \left(xe^x - \int e^x dx\right)$$

$$\boxed{= \frac{3}{2}x^2 - xe^x + e^x + C}$$

4. (10 pts) The region R is bounded by $y = e^x$, $y = 4$, and the y-axis.

   (a) Sketch the region R. Label the curves and all points of intersection of the curves.



   (b) Using either the disks/washers or cylindrical shells method, **set up an integral** to compute the volume generated when R is rotated around the x**-axis**. **State which method you are using.** You do not need to evaluate the integral.

washers

$$V = \pi \int_0^{\ln 4} (4)^2 - (e^x)^2 \, dx = \pi \int_0^{\ln 4} (16 - e^{2x}) \, dx$$

   (c) Using either the disks/washers or cylindrical shells method, **set up an integral** to compute find the volume generated when A is rotated around the y**-axis. State which method you are using.** You do not need to evaluate the integral.

Cylindrical shells

$$V = 2\pi \int_0^{\ln 4} x(4 - e^x) \, dx = 2\pi \int_0^{\ln 4} (4x - xe^x) \, dx$$

5. (5 pts each) Determine whether each series below converges or diverges. Name the test you use and justify your conclusion. (This means show your work!)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{3n+2}}$ Test: alternating series Converge or Diverge

$b_n = \frac{1}{\sqrt{3n+2}}$

① $\lim_{n\to\infty} \frac{1}{\sqrt{3n+2}} = 0$ ✓

② $b_{n+1} = \frac{1}{\sqrt{3n+5}} < \frac{1}{\sqrt{3n+2}} = b_n$ ✓

Since b_n is decreasing and approaches zero, the series converges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+3}{7n^{3/2}+1}$ Test: limit comparison Converge or Diverge

Compare to $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$, a divergent p-series ($p = \frac{1}{2} \leq 1$).

$$\lim_{n\to\infty} \frac{2n+3}{7n^{3/2}+1} \cdot \frac{n^{1/2}}{1} = \lim_{n\to\infty} \frac{2n^{3/2} + 3n^{1/2}}{7n^{3/2}+1} = \frac{2}{7}.$$

Since $\sum \frac{1}{\sqrt{n}}$ diverges and the limit is a number, the original series must also diverge.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ Test: ratio test Converge or Diverge

$$\lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)^3} \cdot \frac{n^3}{2^n} \right| = 2 \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^3 = 2 \cdot 1 = 2 > 1.$$

Since $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, the original series diverges.

(d) $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{1}{3n^2+2}\right)$ Test: divergence test Converge or Diverge

$$\lim_{n\to\infty} \frac{1}{3n^2+2} = 0. \quad \text{So } \lim_{n\to\infty} \cos\left(\frac{1}{3n^2+2}\right) = \cos 0 = 1 \neq 0$$

Since $\displaystyle\lim_{n\to\infty} a_n \neq 0$, the series diverges.

6. (6 pts) Use the Integral Test to determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$ converges or diverges.

$$\int_2^{\infty} \frac{(\ln(x))^{-1/2}}{x}\,dx = \lim_{t\to\infty} \int_2^{t} \frac{(\ln(x))^{-1/2}}{x}\,dx$$

$$= \lim_{t\to\infty} 2(\ln x)^{\frac{1}{2}} \Big|_2^t = \lim_{t\to\infty} \left(2\sqrt{\ln t} - 2\sqrt{\ln(2)}\right)$$

$$= \infty$$

So the series diverges because the integral diverges.

7. (5 pts each) For each power series below, determine the interval of convergence.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n!(x+3)^n}{6^{2n}}$

$$\lim_{n\to\infty}\left|\frac{(n+1)!\,(x+3)^{n+1}}{6^{2n+2}}\cdot\frac{6^{2n}}{n!\,(x+3)^n}\right| = \lim_{n\to\infty} |x+3|\left(\frac{n+1}{6^2}\right) = \infty$$

So the series converges only at $x=-3$.

So the interval of convergence is $[-3,-3]$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-4)^n}{n5^n}$

$$\lim_{n\to\infty}\left|\frac{(x-4)^{n+1}}{(n+1)\,5^{n+1}}\cdot\frac{n\,5^n}{(x-4)^n}\right| = \frac{|x-4|}{5}\left(\lim_{n\to\infty}\frac{n}{n+1}\right) = \frac{|x-4|}{5}$$

We want $\frac{|x-4|}{5} < 1$. So $-5 < x-4 < 5$ or $-1 < x < 9$.

Check endpoints

$x=-1$: $\sum \frac{(-5)^n}{n5^n} = \sum \frac{(-1)^n}{n}$, alt. harmonic

$x=9$: $\sum \frac{5^n}{n5^n} = \sum \frac{1}{n}$, harmonic

Answer

$[-1, 9)$

8. (10 pts) Recall that the Maclaurin series $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

(a) Write $p_4(x)$, the 4th-degree Maclaurin **polynomial** for $\cos(x)$.

$$P_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

(b) Find the Maclaurin series for $f(x) = x\cos(2x)$. Simplify your answer.

$$x\cos(2x) = x\sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n+1}}{n!}$$

(c) Find the Maclaurin series for $G(x) = \int_0^x \cos(\sqrt{t})\, dt$.

$$G(x) = \int_0^x \left(\sum_{n=0}^{\infty} \frac{(-1)^n t^n}{(2n)!}\right) dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)(2n)!}$$

9. (6 pts) Consider the curve $x(t) = t^2 - 3t + 1$, $y(t) = e^t$.

   (a) Determine the slope of the curve at the point $(1, 1)$.

We need the time t so that

$x(t) = t^2 + 3t + 1 = 1$ and

$y(t) = e^t = 0$. So $t = 0$.

Now

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^t}{2t-3}; \quad \text{So } \left.\frac{dy}{dx}\right|_{t=0} = \frac{e^0}{2\cdot 0-3} = \frac{1}{-3} = \frac{-1}{3}$$

   (b) Determine the points where the tangent line is horizontal or vertical, or state that none exist.

horizontal: Need $\frac{dy}{dt} = e^t = 0$. Never.

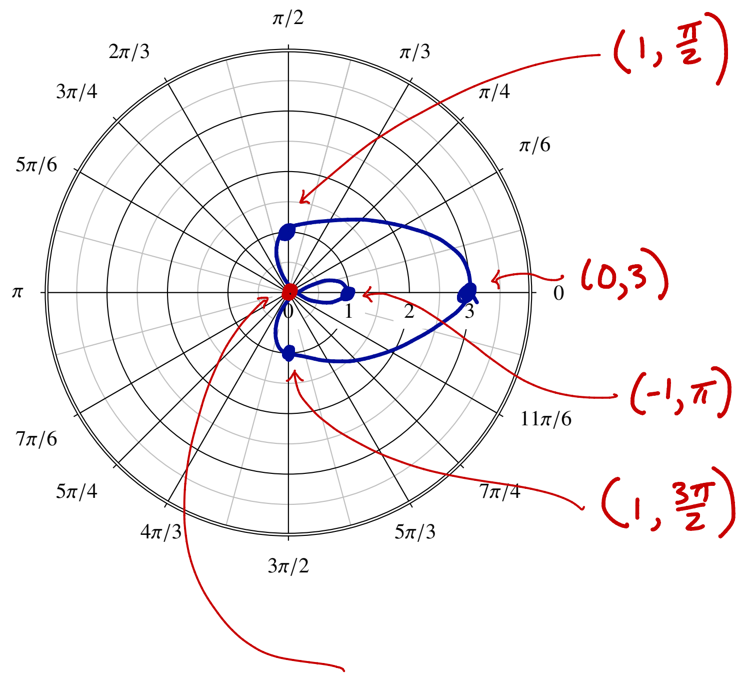
vertical: Need $\frac{dx}{dt} = 2t - 3 = 0$. So $t = 3/2$.

$x(3/2) = (3/2)^2 - 3\left(\frac{3}{2}\right) + 1 = \frac{9}{4} - \frac{9}{2} + 1 = \frac{-5}{4}$

$y(3/2) = e^{3/2}$

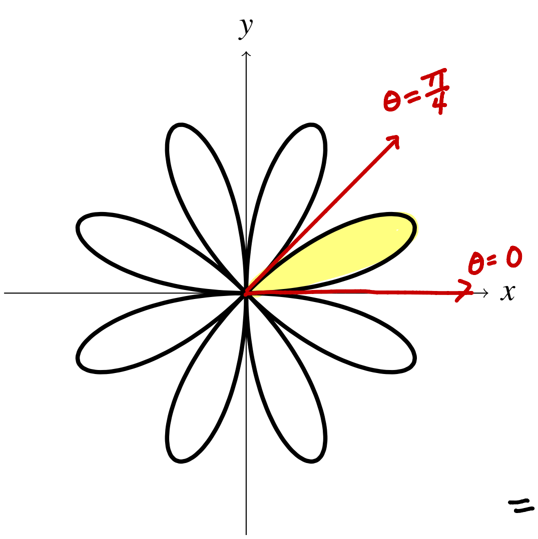
point $\left(\frac{-5}{4}, e^{3/2}\right)$

10. (5 pts) Make a careful sketch of the polar curve $r = 1 + 2\cos(\theta)$. Label at least 4 points.



θ	$r = 1+2\cos\theta$
0	$1+2(1)=3$
$\frac{\pi}{2}$	$1+2\cdot 0=1$
π	-1
$3\pi/2$	1
$\frac{2\pi}{3}$	0
$\frac{5\pi}{3}$	0

11. (6 pts) Compute the area enclosed by the polar curve $r = 3\sin(4\theta)$. The graph of the curve is shown below.



$$A = 8\int_0^{\pi/4} \frac{1}{2}\left(3\sin(4\theta)\right)^2 d\theta$$

$$= 36\int_0^{\pi/4} \sin^2(4\theta)\, d\theta$$

$$= 18\int_0^{\pi/4} \left(1 - \cos(8\theta)\right) d\theta$$

$$= 18\left(\theta - \frac{1}{8}\sin(8\theta)\right)\Big|_0^{\pi/4}$$

$$= 18\left(\left(\frac{\pi}{4} - \frac{1}{8}\sin(2\pi)\right) - \left(0 - \sin 0\right)\right) = \frac{18\pi}{4} = \frac{9}{2}\pi$$

Extra Credit (5 points) Let $f(x) = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{x^n}{3^n}$.

(a) Determine the first 4 terms in the series. (Simplify all coefficients)

$$\binom{\frac{1}{2}}{0} + \binom{\frac{1}{2}}{1}\frac{x}{3} + \binom{1/2}{2}\frac{x^2}{3^2} + \binom{1/2}{3}\frac{x^3}{3^3}$$

$$= 1 + \frac{1}{2}\cdot\frac{x}{3} + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}\frac{x^2}{9} + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}\cdot\frac{x^3}{27}$$

$$= 1 + \frac{1}{6}x - \frac{1}{36}x^2 + \frac{1}{216}x^3$$

(b) The series converges when $x = 2$. Determine the sum.

$$\sum_{n=0}^{\infty} \binom{1/2}{n}\left(\frac{x}{3}\right)^n = \left(1 + \frac{x}{3}\right)^{1/2}$$ where it converges.

So $$\sum_{n=0}^{\infty} \binom{1/2}{n}\left(\frac{2}{3}\right)^n = \left(1 + \frac{2}{3}\right)^{1/2} = \sqrt{\frac{5}{3}}.$$