Math F252 — Midterm I — Fall 2025

Name: Solutions

Rules:

You have 90 minutes to complete this midterm.

Partial credit will be awarded, but you must show your work.

Calculators are not allowed.

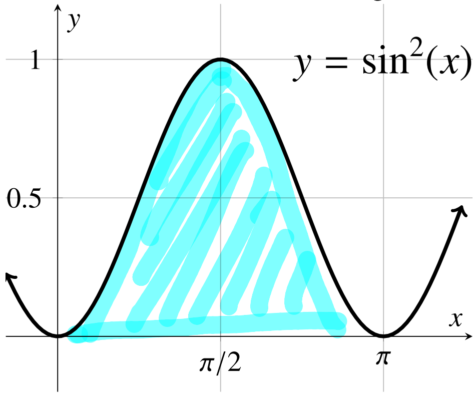
Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question where appropriate.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	12	
2	16	
3	8	
4	11	
5	10	
6	11	
7	32	
Extra Credit	5	
Total	100	

1. (12 points) The region R is bounded by $y = \sin^2(x)$ (graphed below) and the x-axis between $x = 0$ and $x = \pi$. Find the area of region R.

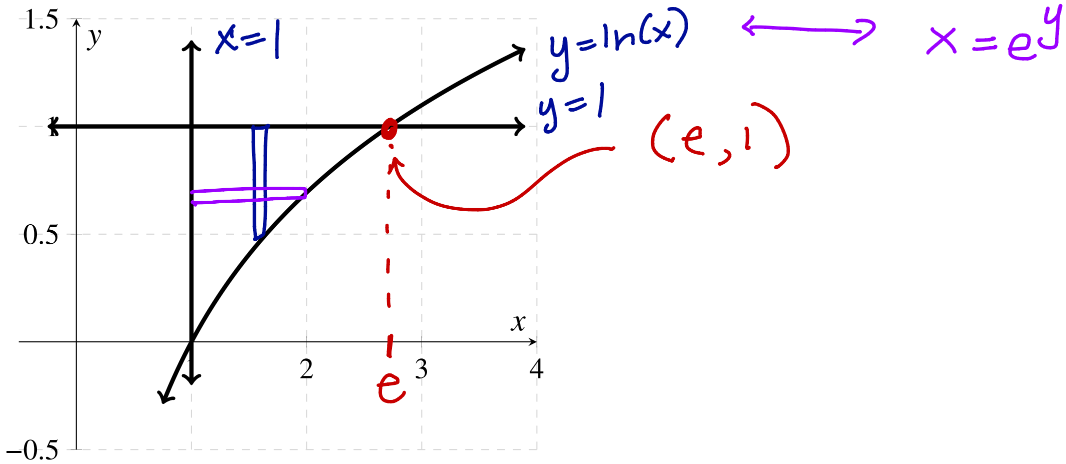


$$\text{area} = \int_0^{\pi} \sin^2(x)\,dx = \frac{1}{2}\int_0^{\pi} \left(1 - \cos(2x)\right) dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right)\Bigg|_0^{\pi} = \frac{1}{2}\left(\left(\pi - \frac{1}{2}\sin(2\pi)\right) - \left(0 - \frac{1}{2}\sin(0)\right)\right)$$

$$\boxed{= \frac{1}{2}\pi}$$

2. (16 points) The region R, sketched below, is bounded by $y = \ln(x)$, $x = 1$ and $y = 1$. **Set up but do not evaluate** a definite integral to calculate each of the volumes described below.



(a) Use the **slicing (disks/washers)** method to find the volume generated when R is rotated about the $\boldsymbol{x}$**-axis.** (Set up only.)

$$V = \int_1^e \pi\left((1)^2 - (\ln x)^2\right)dx = \boxed{\pi\int_1^e \left(1 - (\ln x)^2\right)dx}$$

(b) Use the **shell** method to find the volume generated when R is rotated about the $\boldsymbol{x}$**-axis.** (Set up only.)

$$\boxed{V = \int_0^1 2\pi y\left(e^y - 1\right)dy}$$

(c) Use which ever method you prefer to find the volume generated when R is rotated about the $\boldsymbol{y}$**-axis.** (Set up only.) **State the method you are using.**

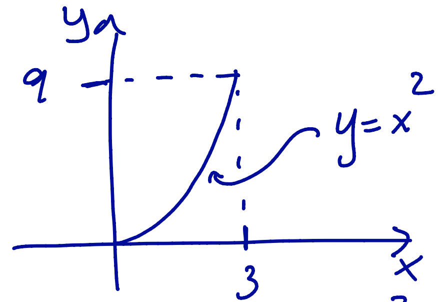
washers

$$V = \int_0^1 \pi\left((e^y)^2 - (1)^2\right)dy = \boxed{\pi\int_0^1 \left(e^{2y} - 1\right)dy}$$

shells

$$V = \boxed{\int_1^e 2\pi x\left(1 - \ln y\right)dy}$$

3. (8 points) Set up but do not evaluate integrals to find the surface area of the volume generated when the curve $y = x^2$ from $x = 0$ to $x = 3$ is rotated about



(a) the x-axis.

$$\boxed{SA = 2\pi \int_0^3 x^2 \sqrt{1 + 4x^2}\, dx}$$

$y' = 2x,\ (y')^2 = 4x^2$

(b) the y-axis.

$$\boxed{SA = 2\pi \int_0^9 \sqrt{y} \sqrt{1 + \frac{1}{4y}}\, dy}$$

$x = \sqrt{y}$

$x' = \frac{1}{2} y^{-1/2}$

$(x')^2 = \frac{1}{4y}$

4. (11 points) A 3-meter long whip antenna has linear density $\rho(x) = 5\ln(x)$ grams per meter for x-values in the interval $[1, 4]$. Determine the mass of the antenna. Include units.

I.B.P

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}\, dx \qquad v = x$

$$\text{mass} = m = \int_1^4 \rho(x)\, dx = \int_1^4 5\ln(x)\, dx$$

$$= 5\left[\left[x \ln x\right]_1^4 - \int_1^4 dx \right] = 5\left[(4\ln 4 - 0) - x\Big]_1^4 \right]$$

$$= 5(4\ln 4 - 4 + 1) = \boxed{5(4\ln 4 - 3) \text{ grams}}$$

5. (10 points)

   (a) If 30 Nm of work are required to compress a spring from 5 meters (neutral length) to 4 meters, determine the spring constant k in Hooke's Law.

$$30 = W = \int_0^1 kx\,dx = \frac{k}{2}x^2\Big|_0^1 = \frac{k}{2}.$$

So $\boxed{k = 60.}$ [So for this spring $F = 60x$]

what we want: displacement $= d$

   (b) For the spring in part a, how far must the spring be compressed from its natural position if the work required to do so is 120 Nm?

$$120 = W = \int_0^d 60x\,dx = 30x^2\Big|_0^d = 30d^2$$

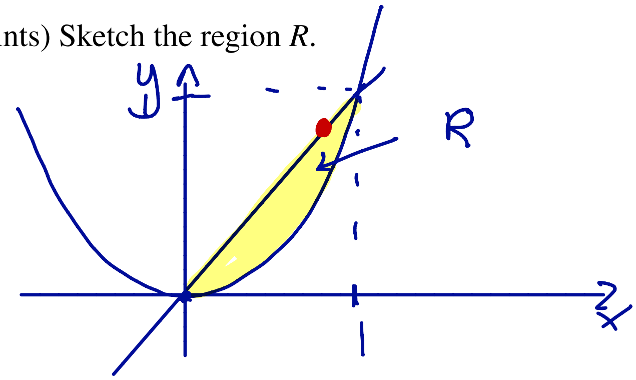
Solve for d: $4 = d^2$

So $d = 2$.

$\boxed{\text{The spring must have been displace 2 meters.}}$

6. (11 points) Let R be the region bounded by $y = x^2$ and $y = x$.

   (a) (3 points) Sketch the region R.



   (b) (6 points) Set up the definite integrals to compute the center of mass (centroid) $(\bar{x}, \bar{y})$ of R. You do not have to evaluate the integrals.

$$\bar{x} = \boxed{\frac{\int_0^1 x(x-x^2)\,dx}{\int_0^1 (x-x^2)\,dx}} \qquad \bar{y} = \boxed{\frac{\frac{1}{2}\int_0^1 (x^2-x^4)\,dx}{\int_0^1 (x-x^2)\,dx}}$$

   (c) (2 points) Suppose someone calculates the center of mass of R to be $(\bar{x}, \bar{y}) = (0.8, 0.8)$. Is this plausible? You can and should answer this **without actually evaluating the integrals in part (b)**.

No. This is not plausible. The point would lie on the line $y=x$. (See red dot) The yellow region will not balance there.

7. (8 points each) Evaluate the indefinite integrals.

(a) $\int \cos^3(x)\sin^3(x)\,dx$ $= \int \cos^3 x \cdot \sin^2 x \cdot \sin x\,dx$

$= \int (\cos^3 x)(1-\cos^2 x)(\sin x\,dx)$

Let $u = \cos x$

$du = -\sin x\,dx$

$= -\int u^3(1-u^2)\,du = \int u^5 - u^3\,du$

$= \frac{1}{6}u^6 - \frac{1}{4}u^4 + C = \boxed{\frac{1}{6}(\cos x)^6 - \frac{1}{4}(\cos x)^4 + C}$

(b) $\int xe^{x/2}\,dx$

IBP

$u = x$ $\quad dv = e^{x/2}\,dx$

$du = dx$ $\quad v = \int e^{x/2}\,dx = 2e^{x/2}$

$= 2xe^{x/2} - \int 2e^{x/2}\,dx = 2xe^{x/2} - 4e^{x/2} + C$

$\boxed{= 2e^{x/2}(x-2) + C}$

(c) $\int \frac{x+4}{x^2+x-6}\,dx = \int \frac{x+4}{(x+3)(x-2)}\,dx = \int \left(\frac{-1/5}{x+3} + \frac{6/5}{x-2} \right) dx$

partial fractions

$\frac{x+4}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$

$= -\frac{1}{5} \ln|x+3| + \frac{6}{5} \ln|x-2| + C$

So $x+4 = A(x-2) + B(x+3)$

Let $x=2$: $6 = B(5)$, $B = 6/5$

$x=-3$: $1 = A(-5)$, $A = -1/5$

(d) $\int \frac{\sqrt{x^2-25}}{x}\,dx = \int \frac{5\tan\theta \cdot 5\sec\theta\tan\theta\,d\theta}{5\sec\theta} = 5\int \tan^2\theta\,d\theta$

Let $x = 5\sec\theta$

$dx = 5\sec\theta\tan\theta$

$\sqrt{x^2-25} = \sqrt{25\sec^2\theta - 25}$

$= 5\tan\theta$

$= 5\int (\sec^2\theta - 1)\,d\theta$

$= 5(\tan\theta - \theta) + C$

$= 5\left(\frac{\sqrt{x^2-25}}{5} - \arctan\left(\frac{\sqrt{x^2-25}}{5} \right) \right) + C$

Extra Credit (5 points) Evaluate the integral $\int x\sin^2(x)\cos(x)\,dx$

$u = x \qquad dv = \sin^2(x)\cos(x)\,dx$

$du = dx \qquad v = \frac{1}{3}\sin^3(x)$

$$= \frac{1}{3}x\sin^3 x - \frac{1}{3}\int \sin^3 x\,dx$$

$$= \frac{1}{3}x\sin^3 x - \frac{1}{3}\int (1-\cos^2 x)\sin x\,dx$$

$u = \cos x$

$du = -\sin x\,dx$

$$= \frac{1}{3}x\sin^3 x + \frac{1}{3}\left(\int (1-u^2)\,du\right)$$

$$= \frac{1}{3}\left(x\sin^3 x + u - \frac{1}{3}u^3\right) + C$$

$$= \frac{1}{3}\left(x\sin^3 x + \cos x - \frac{1}{3}(\cos x)^3\right) + C$$

Half-Angle/Double Angle	Sum and Difference
$\sin^2(x) = \frac{1}{2}(1-\cos(2x))$	$\sin(ax)\cos(bx) = \frac{1}{2}(\sin((a-b)x) + \sin((a+b)x))$
$\cos^2(x) = \frac{1}{2}(1+\cos(2x))$	$\sin(ax)\sin(bx) = \frac{1}{2}(\cos((a-b)x) - \cos((a+b)x))$
$\sin(2x) = 2\sin(x)\cos(x)$	$\cos(ax)\cos(bx) = \frac{1}{2}(\cos((a-b)x) + \cos((a+b)x))$

$$\mathrm{L} = \int_a^b \sqrt{1+(f'(x))^2}\,dx \qquad \mathrm{S} = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$