Math F252 **Midterm 2** **Fall 2025**

Name: Solutions

Rules:

You have 90 minutes to complete this midterm.

Partial credit will be awarded, but you must show your work.

Calculators are not allowed.

One hand-written sheet of notes is allowed.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	8	
2	8	
3	12	
4	8	
5	24	
6	8	
7	12	
8	12	
9	8	
Extra Credit	5	
Total	100	

1. (8 pts) Determine if the integral below converges or diverges. Evaluate the integral if it converges. To earn full points, a solution must contain clear complete work and correct use of notation.

$$\int_0^5 \frac{1}{(5-x)^{2/3}}\,dx = \lim_{t \to 5^-} \int_0^t (5-x)^{-2/3}\,dx = \lim_{t \to 5^-} -3(5-x)^{1/3}\Big|_0^t$$

$$= -3 \lim_{t \to 5^-} \left(\sqrt[3]{5-t} - \sqrt[3]{5}\right) = 3\sqrt[3]{5}$$

2. (8 pts) Consider the sequence $S = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \cdots\right\}$.

   (a) (4 pts) Find a formula for the general term a_n of the sequence assuming that the pattern of the first few terms continues.

   $a_n = \frac{n}{n+1}$ for $n = 1, 2, 3, \ldots$

   (b) (2 pts) Does this sequence converge? Justify your conclusion.

   Yes $\lim_{n \to \infty} \frac{n}{n+1} = 1$

   (c) (2 pts) Does series $\sum_{n=1}^{\infty} a_n$, with terms from the sequence S, converge? Justify your conclusion.

   No. Since $\lim_{n \to \infty} a_n = 1$, by the Divergence Test, the series diverges.

3. (12 pts) For each **convergent** series below, determine its sum.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{2^{3n}}$. $\displaystyle = \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{8^n} = \sum_{n=1}^{\infty} \frac{1}{8}\left(\frac{-3}{8}\right)^{n-1}$

geometric.

$a = \frac{1}{8}$, $r = \frac{-3}{8}$.

So the sum is $\dfrac{\frac{1}{8}}{1-(-\frac{3}{8})} = \frac{1}{8} \cdot \dfrac{1}{1+\frac{3}{8}} = \frac{1}{11}$

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{n^5} - \frac{2}{(n+1)^5}\right)$ $= \left(\frac{2}{1} - \frac{2}{2^5}\right) + \left(\frac{2}{2^5} - \frac{2}{3^5}\right) + \left(\frac{2}{3^5} - \frac{2}{4^5}\right) + \cdots$

Since $S_k = \left(\frac{2}{1} - \frac{2}{2^5}\right) + \cdots + \left(\frac{2}{k^5} - \frac{2}{(k+1)^5}\right)$

$= 2 - \frac{2}{(k+1)^5}$

$\displaystyle\lim_{k\to\infty} \left(2 - \frac{2}{(k+1)^5}\right) = 2$

4. (8 pts) Use the **Integral Test** to determine if the series $\sum_{n=1}^{\infty} ne^{-n^2}$ converges or diverges. (You do not have to verify that the Intergral Test applies.)

$$\int_1^{\infty} xe^{-x^2}\,dx = \lim_{t\to\infty} \int_1^t xe^{-x^2}\,dx = \lim_{t\to\infty} -\frac{1}{2}e^{-x^2}\Big|_1^t$$

$$= \lim_{t\to\infty}\left[-\frac{1}{2}e^{-t^2} + \frac{1}{2}e^{-1}\right] = \lim_{t\to\infty}\left(\frac{-1}{2e^{t^2}} + \frac{1}{2e}\right) = \frac{1}{2e}$$

So the series converges.

5. (6 pts each) For each series below, show whether the series converges or diverges using an appropriate test. **State the test you use.**

(a) $\sum_{n=1}^{\infty} \frac{n!}{(n+3)!}$ $= \sum_{n=1}^{\infty} \frac{1}{(n+3)(n+2)(n+1)}$

Apply L.C.T to $\sum_{n=1}^{\infty} \frac{1}{n^3}$, a convergent p-series.

So $\lim_{n\to\infty} \frac{n^3}{(n+3)(n+2)(n+1)} = 1$.

So the series $\sum_{n=1}^{\infty} \frac{n!}{(n+3)!}$ converges.

(Direct) Comparison test would work, too.

(b) $\sum_{n=1}^{\infty}(-1)^{n+1}\ln\left(\frac{1}{n}\right)$

This diverges by the Divergence Test.

$$\lim_{n\to\infty} \ln\left(\frac{1}{n}\right) = -\infty.$$

So $\lim_{n\to\infty} (-1)^{n+1}\ln\left(\frac{1}{n}\right) = \text{DNE}$

(c) $\sum_{n=1}^{\infty}\frac{n^7}{2^n}$ Converges by the Ratio Test

$$\lim_{n\to\infty} \frac{(n+1)^7}{2^{n+1}}\cdot\frac{2^n}{n^7} = \lim_{n\to\infty} \frac{1}{2}\cdot\left(\frac{n+1}{n}\right)^7 = \frac{1}{2} < 1$$

So, the series $\sum \frac{n^7}{2^n}$ converges.

(Root Test also a good choice, here)

(d) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt[3]{n+3}}$

Converges by the Alternating Series Test.

$b_n = \dfrac{1}{\sqrt[3]{n+3}}$.

① $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt[3]{2n}} = 0$ ✓

② $\sqrt[3]{n+4} > \sqrt[3]{n+3}$

So $\dfrac{1}{\sqrt[3]{n+4}} < \dfrac{1}{\sqrt[3]{n+3}}$

So $b_{n+1} < b_n$ ✓

6. (8 pts) Determine whether the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n^3}{5+n^5}$ is **absolutely convergent, conditionally convergent, or divergent.**

It's absolutely convergent.

$\displaystyle\sum_{n=1}^{\infty}\left|(-1)^{n+1}\left(\frac{n^3}{5+n^5}\right)\right| = \sum_{n=1}^{\infty}\frac{n^3}{5+n^5}$. This series

converges by the L.C.T with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$, a convergent

p-series.

$\displaystyle\lim_{n\to\infty}\frac{n^3}{5+n^5}\cdot\frac{n^2}{1} = 1.$

7. (12 pts) Determine the radius of convergence, R, and the interval of convergence for each power series below.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} |x| \cdot \frac{1}{n+1} = 0 < 1, \text{ always.}$$

So $R = \infty$ and I.O.C is $(-\infty, \infty)$

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{5n+4}$

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{5n+9} \cdot \frac{5n+4}{(x-2)^n} \right| = \lim_{n\to\infty} |x-2| \left(\frac{5n+4}{5n+9} \right) = |x-2| < 1$$

So $R = 1$. The series converges $(+1, 3)$.

Check end points

$x=3$: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{5n+4}$. This diverges by L.C.T with $\sum \frac{1}{n}$, divergent p-series.

$$\lim_{n\to\infty} \frac{n}{5n+4} = \frac{1}{5} \checkmark$$

$x=1$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{5n+4}$. This converges by A.S.T.

① $\displaystyle\lim_{n\to\infty} \frac{1}{5n+4} = 0$ and ② $b_{n+1} = \dfrac{1}{5n+9} < \dfrac{1}{5n+4} = b_n$

ANSWER: $[1, 3)$ is the interval of convergence

8. (12 pts)

(a) Find a power series representation for the function $f(x) = \dfrac{x}{1-4x}$.

$$f(x) = x\sum_{n=0}^{\infty}(4x)^n = \sum_{n=0}^{\infty} 4^n x^{n+1}$$

(b) Determine the radius of convergence and interval of convergence for the power series in part (a).

$$\lim_{n\to\infty}\left|\frac{4^{n+1}x^{n+2}}{4^n x^{n+1}}\right| = \lim_{n\to\infty} 4|x| < 1 \text{ . So } |x| < \frac{1}{4}$$

$R = \frac{1}{4}$, I.O.C $\left(-\frac{1}{4}, \frac{1}{4}\right)$ $\displaystyle\sum_{n=0}^{\infty} 1^n$ and $\displaystyle\sum_{n=0}^{\infty} -(-1)^n$

are both divergent geometric series (w/ $|r| = 1 \geq 1$)

(c) Use your answer from part (a) to find a power series representation for $f'(x)$.

$$f'(x) = \sum_{n=0}^{\infty} 4^n (n+1) x^n$$

9. (8 pts) Find the Taylor series for $f(x) = \frac{1}{x}$ at $a = 2$. Your answer should be simplified.

$f(x) = x^{-1}$

$f'(x) = -x^{-2}$

$f''(x) = 1 \cdot 2 \cdot x^{-3}$

$f'''(x) = {}^{-}1 \cdot 2 \cdot 3 x^{-4}$

$\vdots$

$f^{(n)}(x) = (-1)^{n-1} \cdot n! \, x^{-(n+1)}$

$f^{(n)}(2) = (-1)^{n-1} (n!) \cdot 2^{-(n+1)}$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} n!}{n! \, 2^{n+1}} (x-2)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (x-2)^n}{2^{n+1}}$$

Extra Credit (5 pts) Determine the convergence of the two series below.

a. $\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)^n$ Converges by the root test.

$$\lim_{n\to\infty} \sqrt[n]{(\sqrt[n]{2} - 1)^n} = \lim_{n\to\infty} \sqrt[n]{2} - 1 = 1 - 1 = 0 < 1 \checkmark$$

b. $\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)$

Diverges by L.C.T. to $\sum_{n=1}^{\infty} \frac{1}{n}$, divergent p-series

$$\lim_{n\to\infty} \frac{\sqrt[n]{2} - 1}{\frac{1}{n}} = \lim_{n\to\infty} \frac{2^{\frac{1}{n}} - 1}{\frac{1}{n}} \overset{H}{=} \lim_{n\to\infty} \frac{\ln(2) \cdot 2^{\frac{1}{n}} \cdot (-1) n^{-2}}{(-1) n^{-2}} = \lim_{n\to\infty} \ln(2) \, 2^{\frac{1}{n}}$$

$\uparrow$ form $\frac{0}{0}$

$= \ln(2) \checkmark$