Name: SOLUTIONS

Math 252 Calculus 2 (Bueler) Thursday, 17 February 2022

Midterm Exam 1

No book, notes, electronics, calculator, or internet access. 100 points possible. 70 minutes maximum.

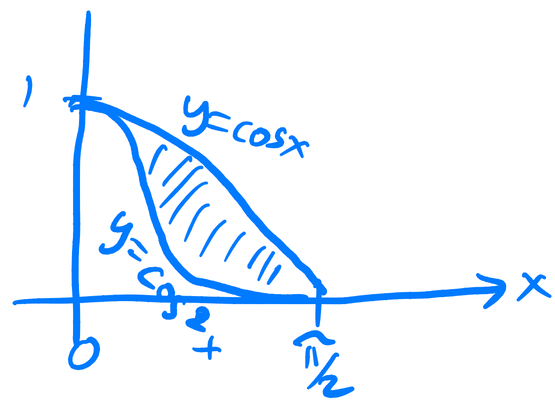
1. (*8 pts*) Compute the area between the curves $y = \cos(x)$ and $y = \cos^2(x)$ on the interval $0 \le x \le \pi/2$. (*Hint. Be careful about which curve is above the other.*)

$$A = \int_0^{\pi/2} \cos x - \cos^2 x \, dx$$

$$= \int_0^{\pi/2} \cos x - \tfrac{1}{2}(1 + \cos 2x)\, dx$$

$$= \left[\sin x - \tfrac{1}{2}x - \tfrac{1}{4}\sin(2x)\right]_0^{\pi/2}$$

$$= \left(1 - \tfrac{\pi}{4} - 0\right) - (0) = 1 - \frac{\pi}{4}$$



2. (*6 pts*) Completely set up, but do not evaluate, a definite integral for the length of the curve $y = \dfrac{1}{x}$ on the interval $x = 1$ to $x = 10$.

$$f(x) = \frac{1}{x}$$

$$f'(x) = -x^{-2}$$

$$L = \int_1^{10} \sqrt{1 + x^{-4}}\, dx$$

3. Evaluate and simplify the following indefinite and definite integrals.

(a) (*6 pts*) $\int \tan x \, dx =$

$$\int \frac{\sin x}{\cos x}\,dx = \int \frac{-du}{u}$$

$$\begin{pmatrix} u = \cos x \\ du = -\sin x \, dx \end{pmatrix}$$

$$= -\ln|u| + C = -\ln|\cos x| + C$$

(b) (*6 pts*) $\int_0^1 3^x \, dx =$

$$\left[\frac{3^x}{\ln 3}\right]_0^1 = \frac{3-1}{\ln 3} = \frac{2}{\ln 3}$$

(c) (*6 pts*) $\int_0^{\pi/4} \tan^3 x \sec^2 x \, dx =$

$$\int_0^1 u^3 \, du$$

$$\begin{pmatrix} u = \tan x \\ du = \sec^2 x \, dx \end{pmatrix}$$

$$= \left.\frac{u^4}{4}\right]_0^1 = \frac{1}{4}$$

(d) (*8 pts*) $\int \cos^2(7t)\sin^3(7t)\,dt =$ $\int \cos^2(7t)\sin^2(7t)\sin(7t)\,dt$

$$= \int \cos^2(7t)\left(1-\cos^2(7t)\right)\sin(7t)\,dt$$

$(u = \cos(7t),\quad du = -7\sin(7t)\,dt)$

$$= \int u^2(1-u^2)\,\frac{du}{-7} = -\frac{1}{7}\int u^2 - u^4\,du$$

$$= -\frac{1}{7}\left(\frac{u^3}{3} - \frac{u^5}{5}\right) + C = \frac{1}{35}\cos^5(7t) - \frac{1}{21}\cos^3(7t) + C$$

(e) (*8 pts*) $\int \cos(7t)\cos(3t)\,dt =$

← see $\cos(ax)\cos(bx)$ on last page

$$= \frac{1}{2}\int \cos(4t) + \cos(10t)\,dt$$

$$= \frac{1}{2}\left(\frac{1}{4}\sin(4t) + \frac{1}{10}\sin(10t)\right) + C$$

$$= \frac{1}{8}\sin(4t) + \frac{1}{20}\sin(10t) + C$$

(f) (*8 pts*) $\displaystyle\int \frac{x}{x^2-4x-5}\,dx =$

$$\frac{x}{(x-5)(x+1)} = \frac{A}{x-5} + \frac{B}{x+1}$$

$$1\cdot x + 0 = A(x+1) + B(x-5)$$

$$= (A+B)x + (A-5B)$$

$$\left.\begin{array}{l} A+B=1 \\ A-5B=0 \end{array}\right\} 6B=1 \quad \therefore B=\frac{1}{6}$$

$$A=\frac{5}{6}$$

$$= \int \frac{5/6}{x-5} + \frac{1/6}{x+1}\,dx$$

$$= \frac{5}{6}\ln|x-5| + \frac{1}{6}\ln|x+1| + C$$

(g) (*8 pts*) $\displaystyle\int z^2\,e^z\,dz = z^2e^z - 2\int z e^z\,dz$

$$\left(\begin{array}{ll} u=z^2 & v=e^z \\ du=2z\,dz & dv=e^z dz \end{array}\right)$$

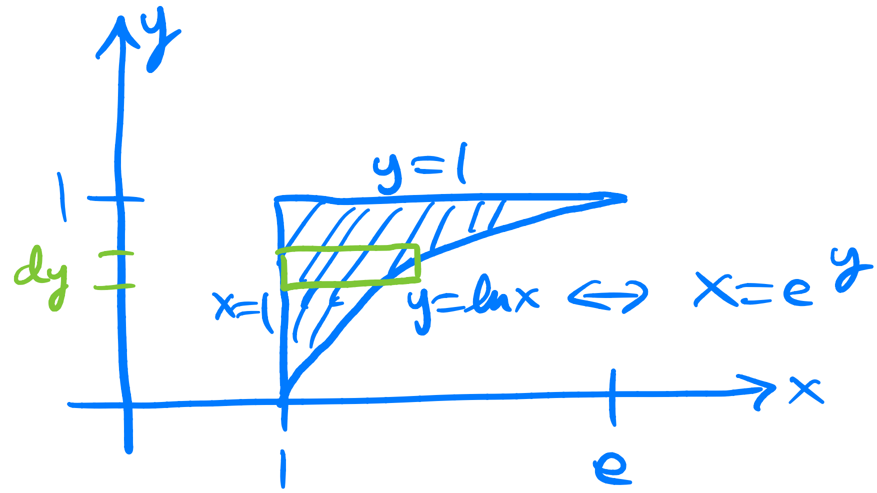
$$\left(\begin{array}{ll} w=z & y=e^z \\ dw=dz & dy=e^z dz \end{array}\right)$$

$$= z^2e^z - 2\left(ze^z - \int e^z\,dz\right)$$

$$= z^2e^z - 2ze^z + 2e^z + C$$

$$= (z^2-2z+2)e^z + C$$

4. **(a)** (*4 pts*) Sketch the region bounded by the curves $y = \ln x$, $x = 1$, and $y = 1$.



(b) (*8 pts*) Use shells to find the volume of the solid of revolution found by rotating the region in part **(a)** around the x-axis.

$$V = \int_0^1 2\pi y \cdot (e^y - 1) \cdot dy$$

$$= 2\pi \int_0^1 y e^y \, dy - 2\pi \int_0^1 y \, dy$$

$$= 2\pi \left(\left[y e^y \right]_0^1 - \int_0^1 e^y \, dy \right) - 2\pi \cdot \frac{1}{2}$$

$$\left(\begin{array}{ll} u = y & v = e^y \\ du = dy & dv = e^y dy \end{array} \right)$$

$$= 2\pi \left(e - 0 - \left[e^y \right]_0^1 \right) - \pi$$

$$= 2\pi (e - e + 1) - \pi = 2\pi - \pi = \pi$$

(c) (*8 pts*) Fully set up, but do not evaluate, the three integrals needed to compute the center of mass $(\bar{x}, \bar{y})$ of the region in part **(a)** (previous page). Then fill in the blanks at the bottom, to show how to compute the values $\bar{x}$ and $\bar{y}$.

either is correct

$$m = \int_1^e 1 - \ln x \, dx = \int_0^1 e^y - 1 \, dy$$

$$M_y = \int_1^e x(1 - \ln x)\, dx = \int_0^1 \tfrac{1}{2}(e^y + 1)(e^y - 1)\, dy$$

$$M_x = \int_1^e \tfrac{1}{2}(1 + \ln x)(1 - \ln x)\, dx$$
$$= \int_0^1 y(e^y - 1)\, dy$$

$$\bar{x} = \frac{\boxed{M_y}}{\boxed{m}}, \qquad \bar{y} = \frac{\boxed{M_x}}{\boxed{m}}$$

5. Which trigonometric substitution would you use for the following two integrals? Write the substitution in the box. (There is no need to compute the integrals here.)

(a) (*4 pts*) $\displaystyle\int \sqrt{x^2 - 16}\, dx$

$$\boxed{x = 4\sec\theta}$$

(b) (*4 pts*) $\displaystyle\int \frac{t^2}{\sqrt{1-4t^2}}\, dt$

$$\boxed{2t = \sin\theta}$$

6. (*8 pts*) Evaluate and simplify the integral in **5(b)** above.

$$2\,dt = \cos\theta\, d\theta$$

$$t = \tfrac{1}{2}\sin\theta$$

$$\sqrt{1-4t^2} = \cos\theta$$

$$= \int \frac{\frac{1}{4}\sin^2\theta}{\cancel{\cos\theta}} \frac{1}{2}\cancel{\cos\theta}\, d\theta$$

$$= \frac{1}{8}\int \sin^2\theta\, d\theta$$

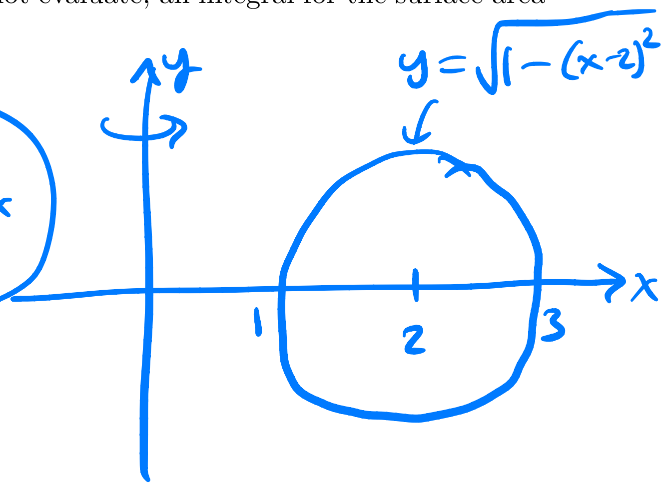
$$= \frac{1}{16}\int 1 - \cos(2\theta)\, d\theta = \frac{1}{16}\left(\theta - \frac{1}{2}\sin(2\theta)\right) + C$$

$$= \frac{1}{16}\left(\arcsin(2t) - \sin\theta\cos\theta\right) + C$$

$$= \frac{1}{16}\left(\arcsin(2t) - 2t\sqrt{1-4t^2}\right) + C$$

Extra Credit. (*3 pts*) A donut (torus) surface is created by rotating a circle with radius one and center $(x, y) = (2, 0)$ around the y-axis. Fully set up, but do not evaluate, an integral for the surface area of this donut.

$$S = 2\int_1^3 2\pi x \sqrt{1 + \frac{(2-x)^2}{1-(x-2)^2}}\, dx$$



$$f(x) = (1-(x-2)^2)^{1/2}$$

$$f'(x) = \tfrac{1}{2}(1-(x-2)^2)^{-1/2}(-2(x-2))$$

$$= \frac{2-x}{\sqrt{1-(x-2)^2}}$$

it turns out that the surface area of a torus

is $4\pi^2 ab$, and above integral gives 8π

You may find the following **trigonometric formulas** useful. However, there are other trig. formulas, not listed here, which you should have in memory, or which you know how to derive from these.

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$$
$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$$
$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$$