Name:________________________________

Math 252 Calculus 2 (Bueler) Thursday, 7 April 2022

Midterm Exam 2

No book, notes, electronics, calculator, or internet access. 100 points possible. 70 minutes maximum.

1. Compute and simplify the improper integrals, or show they diverge. Use correct limit notation.

(a) (*5 pts*) $\displaystyle \int_0^\infty \frac{x^2\,dx}{1+x^3} =$

(b) (*5 pts*) $\displaystyle \int_0^1 \frac{dx}{\sqrt[4]{x}} =$

2. (*5 pts*) Does the following series converge or diverge? Show your work, including naming any test you use. (*Hint.* Previous problem? Or another test?)

$$\sum_{n=0}^{\infty} \frac{n^2}{1+n^3} =$$

3. Do the following series converge or diverge? Show your work, including naming any test you use.

(a) *(5 pts)* $\displaystyle\sum_{n=0}^{\infty} \frac{n+2}{4^n}$

(b) *(5 pts)* $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{4^n}$

(c) *(5 pts)* $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

(d) (*5 pts*) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(e) (*5 pts*) $\displaystyle\sum_{n=1}^{\infty} \frac{(\cos n)^2}{n^2}$

4. (*5 pts*) For one of the five series in problem **3**, it is possible to compute the value of the infinite series. Which one? Explain why, and compute the value.

5. Consider the infinite series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$

(a) (*5 pts*) Write the series using sigma (Σ) notation.

(b) (*5 pts*) Compute and simplify S_4, the partial sum of the first four terms.

6. (*5 pts*) Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ converge absolutely, conditionally, or neither (diverge)? Show your work and circle one answer.

CONVERGES ABSOLUTELY   CONVERGES CONDITIONALLY   DIVERGES

7. By any method, write a power series for the following functions. Show your work.

(a) (*5 pts*) $\dfrac{1}{1+2x}$

(b) (*7 pts*) $\arctan x$

(*Hint.* Integrate a series derived from the geometric series.)

8. Find the interval of convergence of the following power series.

(a) (*7 pts*) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n2^n}$

(b) (*5 pts*) $\displaystyle\sum_{n=1}^{\infty} (n-1)!\, x^n$

9. (*9 pts*) Find the Taylor series of $f(x) = \dfrac{1}{x}$ at basepoint $a = 2$.

10. (*7 pts*) Use the midpoint rule with $n = 2$ subintervals to estimate $\int_0^1 x^3\,dx$. Simplify your result.

Extra Credit. (*3 pts*) Recall the famous Maclaurin series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Suppose I put $x = -\frac{1}{2}$ in this power series. In fact, suppose I compute the 10th partial sum $S_{10} = \sum_{n=0}^{10} \frac{(-1/2)^n}{n!}$. I observe, using Matlab, that it gives more than 10 digits of accuracy in approximating $e^{-1/2} = \frac{1}{\sqrt{e}}$. Explain why, using a known fact about remainders.

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Summary of Convergence Tests

Series or Test	Conclusions	Comments
Divergence Test For any series $\sum_{n=1}^{\infty} a_n$, evaluate $\lim_{n\to\infty} a_n$.	If $\lim_{n\to\infty} a_n = 0$, the test is inconclusive.	This test cannot prove convergence of a series.
	If $\lim_{n\to\infty} a_n \neq 0$, the series diverges.	
Geometric Series $\sum_{n=1}^{\infty} ar^{n-1}$	If $\lvert r\rvert < 1$, the series converges to $a/(1-r)$.	Any geometric series can be reindexed to be written in the form $a + ar + ar^2 + \cdots$, where a is the initial term and r is the ratio.
	If $\lvert r\rvert \geq 1$, the series diverges.	
***p*-Series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$	If $p > 1$, the series converges.	For $p = 1$, we have the harmonic series $\sum_{n=1}^{\infty} 1/n$.
	If $p \leq 1$, the series diverges.	
Comparison Test For $\sum_{n=1}^{\infty} a_n$ with nonnegative terms, compare with a known series $\sum_{n=1}^{\infty} b_n$.	If $a_n \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.	Typically used for a series similar to a geometric or p-series. It can sometimes be difficult to find an appropriate series.
	If $a_n \geq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	
Limit Comparison Test For $\sum_{n=1}^{\infty} a_n$ with positive terms, compare with a series $\sum_{n=1}^{\infty} b_n$ by evaluating $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.	If L is a real number and $L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.	Typically used for a series similar to a geometric or p-series. Often easier to apply than the comparison test.

Series or Test	Conclusions	Comments
	If $L=0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.	
	If $L=\infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	
Integral Test If there exists a positive, continuous, decreasing function f such that $a_n = f(n)$ for all $n \geq N$, evaluate $\int_N^{\infty} f(x)dx.$	$\int_N^{\infty} f(x)dx$ and $\sum_{n=1}^{\infty} a_n$ both converge or both diverge.	Limited to those series for which the corresponding function f can be easily integrated.
Alternating Series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ or $\sum_{n=1}^{\infty} (-1)^n b_n$	If $b_{n+1} \leq b_n$ for all $n \geq 1$ and $b_n \to 0$, then the series converges.	Only applies to alternating series.
Ratio Test For any series $\sum_{n=1}^{\infty} a_n$ with nonzero terms, let $\rho = \lim_{n \to \infty} \left\|\frac{a_{n+1}}{a_n}\right\|.$	If $0 \leq \rho < 1$, the series converges absolutely.	Often used for series involving factorials or exponentials.
	If $\rho > 1$ or $\rho = \infty$, the series diverges.	
	If $\rho = 1$, the test is inconclusive.	
Root Test For any series $\sum_{n=1}^{\infty} a_n$, let $\rho = \lim_{n \to \infty} \sqrt[n]{\|a_n\|}.$	If $0 \leq \rho < 1$, the series converges absolutely.	Often used for series where $\|a_n\| = b_n^n.$
	If $\rho > 1$ or $\rho = \infty$, the series diverges.	
	If $\rho = 1$, the test is inconclusive.	