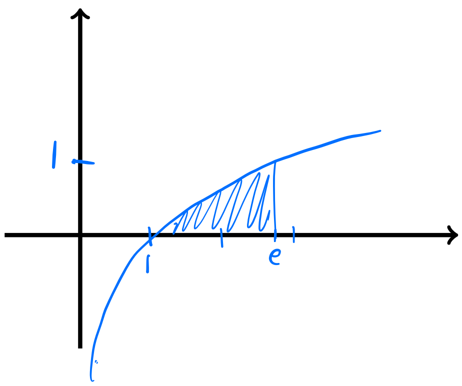
1. (*6 pts*) Sketch the area under the curve $f(x) = \ln x$ between $x = 1$ and $x = e$. Find this area by evaluating an integral.



$u = \ln x \quad dv = dx$

$du = \frac{1}{x}\,dx \quad v = x$

$$\int_1^e \ln x \, dx$$

$$= \left[x \ln x\right]_1^e - \int_1^e x \cdot \left(\tfrac{1}{x}\right) dx$$

$$= \left[e \cdot 1 - 0\right] - \left[x\right]_1^e$$

$$= e - (e-1)$$

$$\boxed{= 1}$$

2. (*6 pts*) Evaluate the sum $\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}}$.

Series?

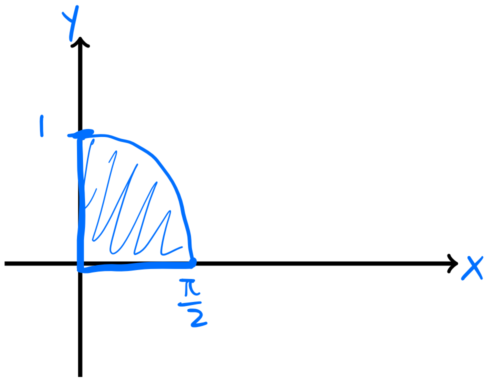
$$= \frac{1}{3} \cdot \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n$$

$$= \frac{1}{3} \cdot \frac{1}{1-\left(-\frac{1}{3}\right)}$$

$$= \frac{1}{3} \cdot \frac{1}{\frac{4}{3}}$$

$$= \frac{1}{3} \cdot \frac{3}{4} = \boxed{\frac{1}{4}}$$

3. **(a)** (*3 pts*) On the axes below, sketch the region R in the first quadrant bounded by $y = \cos(x)$, $y = 0$, and $x = 0$.



(b) (*6 pts*) Use an integral to find the volume of the solid obtained by rotating R about the x-axis.

$$\int_0^{\pi/2} \pi \cdot \cos^2 x \, dx = \pi \int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right) dx$$

$$= \pi \left[\frac{1}{2}x + \frac{1}{4}\sin(2x)\right]_0^{\pi/2}$$

$$= \pi \left[\frac{\pi}{4} + \frac{1}{4}\cdot\sin(\pi) - 0\right]$$

$$= \boxed{\frac{\pi^2}{4}}$$

(c) (*5 pts*) Set up, but do not evaluate, an integral for the surface area of the solid in part **(b)**.

$$\int_0^{\pi/2} 2\pi \cdot y \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

$$\frac{dy}{dx} = \frac{d}{dx}\cos(x) = -\sin x$$

$$\boxed{= \int_0^{\pi/2} 2\pi \cos x \sqrt{1 + \sin^2 x} \, dx}$$

4. Evaluate the indefinite integrals below.

(a) (*6 pts*) $\int \cos^5(\theta)\, d\theta$

$$= \int \cos(\theta)\left(1-\sin^2(\theta)\right)^2 d\theta$$

$$= \int \cos\theta \left(1 - 2\sin^2\theta + \sin^4\theta\right) d\theta$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$$= \int \left(1 - 2u^2 + u^4\right) du$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C$$

$$\boxed{= \sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta + C}$$

(b) (*6 pts*) $\int \frac{1}{(4+x^2)^{3/2}}\, dx$

$$= \int \frac{1}{4^{3/2}\cdot\left(1+\left(\frac{x}{2}\right)^2\right)^{3/2}} dx$$

$\tan\theta = \frac{x}{2}$

$1 + \left(\frac{x}{2}\right)^2 = 1 + \tan^2\theta = \sec^2\theta$

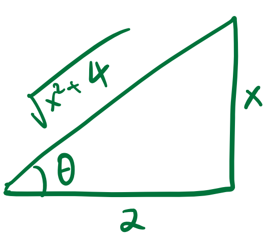
$dx = 2\cdot\sec^2\theta\, d\theta$

$$= \frac{1}{8}\cdot\int \frac{1}{\sec^3\theta}\cdot 2\sec^2\theta\, d\theta$$

$$= \frac{1}{4}\cdot\int \frac{1}{\sec\theta}\, d\theta$$

$$= \frac{1}{4}\int \cos\theta\, d\theta$$

$$= \frac{1}{4}\sin\theta + C$$



$\sin\theta = \frac{x}{\sqrt{x^2+4}}$

$$\boxed{= \frac{x}{4\sqrt{x^2+4}} + C}$$

5. (*6 pts*) A spring with a relaxed length of 2 meters requires 3 Newtons force to stretch to a length of 2.1 meters. How much work would it take to stretch the spring from 2 meters to 2.4 meters?

$F(x) = kx,\quad F(0.1) = 3\text{ N} \to 3 = k(0.1) \to k = 30$

$$\int_0^{0.4} 30x\,dx = \left[15x^2\right]_0^{0.4} = 15\cdot\left(\tfrac{2}{5}\right)^2 = 15\cdot\frac{4}{25}$$

$$= 3\cdot\frac{4}{5}$$

$$\boxed{= \frac{12}{5}\text{ J}}$$

$$\left(= 2.4\text{ J}\right)$$

6. (*6 pts*) Find the Maclaurin series of $\sinh x = \dfrac{e^x - e^{-x}}{2}$. (one series?)

$$\sinh x = \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!} - \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\right)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{1}{n!}x^n - \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n\right)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{n!} - \frac{(-1)^n}{n!}\right)x^n$$

$$(1-(-1)^n) = \begin{cases} 0 & \text{if } n \text{ even} \\ 2 & \text{if } n \text{ odd}\end{cases}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{2}{(2n+1)!}x^{2n+1}$$

$$= \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}x^{2n+1}$$

7. (*5 pts each*) Do the following series converge or diverge? Show your work; **state the name** of any test you use.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{2n+1}}$

Alt. Series Test :

$b_n = \frac{1}{\sqrt{2n+1}} > 0$, decreasing,

$\lim_{n\to\infty} \frac{1}{\sqrt{2n+1}} = 0$

$\rightarrow$ series $\boxed{\text{converges}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n+1}}{n^n}$

Root test :

$\lim_{n\to\infty} \sqrt[n]{\frac{4^{n+1}}{n^n}} = \lim_{n\to\infty} \sqrt[n]{4} \cdot \left(\frac{4}{n}\right) = 1 \cdot 0 = 0$

$\rightarrow$ series $\boxed{\text{converges}}$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$

Comparison test with $\sum_{n=2}^{\infty} \frac{1}{n^2}$, which converges (p-series, $p = 2 > 1$)

For $n \geq 3$, $\ln n > 1$,

so $\frac{1}{n^2 \ln n} < \frac{1}{n^2}$.

$\rightarrow$ series $\boxed{\text{converges}}$

8. (*6 pts each*) For each power series, determine the **interval** of convergence.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{n}$

Ratio test: $\displaystyle\lim_{n\to\infty} \left| \frac{(x+2)^{n+1}}{n+1} \cdot \frac{n}{(x+2)^n} \right|$

$\displaystyle = \lim_{n\to\infty} \left| (x+2) \cdot \frac{n}{n+1} \right| = |x+2|$

converges when $|x+2| < 1$, i.e. $x \in (-3, -1)$.

Endpoints: $x = -3$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-3+2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges (alt. harmonic series)

$x = -1$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1+2)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series)

Interval of convergence: $\boxed{[-3, -1)}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{8(n+1)!}$

Ratio test: $\displaystyle\lim_{n\to\infty} \left| \frac{x^{2n+2}}{8(n+2)!} \cdot \frac{8(n+1)!}{x^{2n}} \right|$

$\displaystyle = \lim_{n\to\infty} \left| \frac{x^2}{n+2} \right| = 0$

$\rightarrow$ converges for every x.

Interval of convergence: $\boxed{\mathbb{R}}$

9. Let $f(x) = \dfrac{1}{1+x}$.

(a) (*4 pts*) Find a formula for $f^{(n)}(x)$, the nth derivative of $f(x)$.

$f^{(0)}(x) = f(x) = (1+x)^{-1}$

$f^{(1)}(x) = (-1)(1+x)^{-2}$

$f^{(2)}(x) = (-1)(-2)(1+x)^{-3}$

$\vdots$

$$f^{(n)}(x) = (-1)(-2)\cdots(-n)(1+x)^{-(n+1)}$$

$$\boxed{= (-1)^n \cdot n! \cdot (1+x)^{-n-1}}$$

(b) (*5 pts*) Find the Taylor series for $f(x)$ centered at $a = 1$. Your answer should be reasonably simplified.

$$f^{(n)}(a) = (-1)^n n! (1+1)^{-n-1} = \frac{(-1)^n n!}{2^{n+1}}$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} \cdot (x-1)^n$$

$$= \sum_{n=0}^{\infty} \frac{(x-1)^n}{2 \cdot (-2)^n}$$

10. Consider the curve defined by the parametric equations $x = \ln(t), \quad y = (t-1)^2.$

(a) (*4 pts*) Find the equation of the tangent line at $t = 2$.

$\frac{dy}{dt} = 2(t-1)$

$\frac{dx}{dt} = \frac{1}{t}$

$\frac{dy}{dx} = \frac{2(t-1)}{\left(\frac{1}{t}\right)} = 2t(t-1)$

slope at $t=2$: $2(2)(2-1) = 4$

point at $t=2$: $(\ln(2), (2-1)^2) = (\ln 2, 1)$

Tangent line :

$$\boxed{y - 1 = 4(x - \ln 2)}$$

(b) (*3 pts*) For $t > 0$, determine the points on the curve—give the x, y coordinates of these points—at which the tangent line is horizontal.

Set $\frac{dy}{dx} = 0 \rightarrow 2t(t-1) = 0 \rightarrow t = 0, 1$.

Only positive solution is $t = 1$.

Point at $t=1$: $x = \ln(1) = 0$, $y = (1-1)^2 = 0$ $\rightarrow \boxed{(x,y) = (0,0)}$.

(c) (*4 pts*) Set up but do not evaluate an integral for the length of the curve from $t = 1$ to $t = 2$.

$$\int_1^2 \sqrt{\left(\frac{1}{t}\right)^2 + \left(2(t-1)\right)^2} \, dt$$

11. **(a)** (*5 pts*) Find the Maclaurin series for e^{-x^2}. (**Hint.** Start from a known Maclaurin series.)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \rightarrow \quad e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$

$$\boxed{= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}}$$

(b) (*5 pts*) Compute an infinite series for the integral $\int_0^3 e^{-x^2}\,dx$.

$$\int_0^3 \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\right) dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1}\right]_0^3$$

$$\boxed{= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^{2n+1}}{n!\,(2n+1)}}$$

(c) (*5 pts*) Show that the infinite series in part **(b)** converges. **State the name** of any test you use.

Ratio test on $\sum_{n=0}^{\infty} \frac{3^{2n+1}}{n!\,(2n+1)}$: $\lim_{n\to\infty} \left| \frac{3^{2n+3}}{(n+1)!\,(2n+3)} \cdot \frac{n!\,(2n+1)}{3^{2n+1}} \right|$

$$= \lim_{n\to\infty} \frac{9}{n+1} \cdot \frac{2n+1}{2n+3} = 0$$

→ Series converges (absolutely)

12. **(a)** (*4 pts*) Convert the polar equation $r = \cos\theta$ to rectangular form.

$$r^2 = r\cos\theta$$

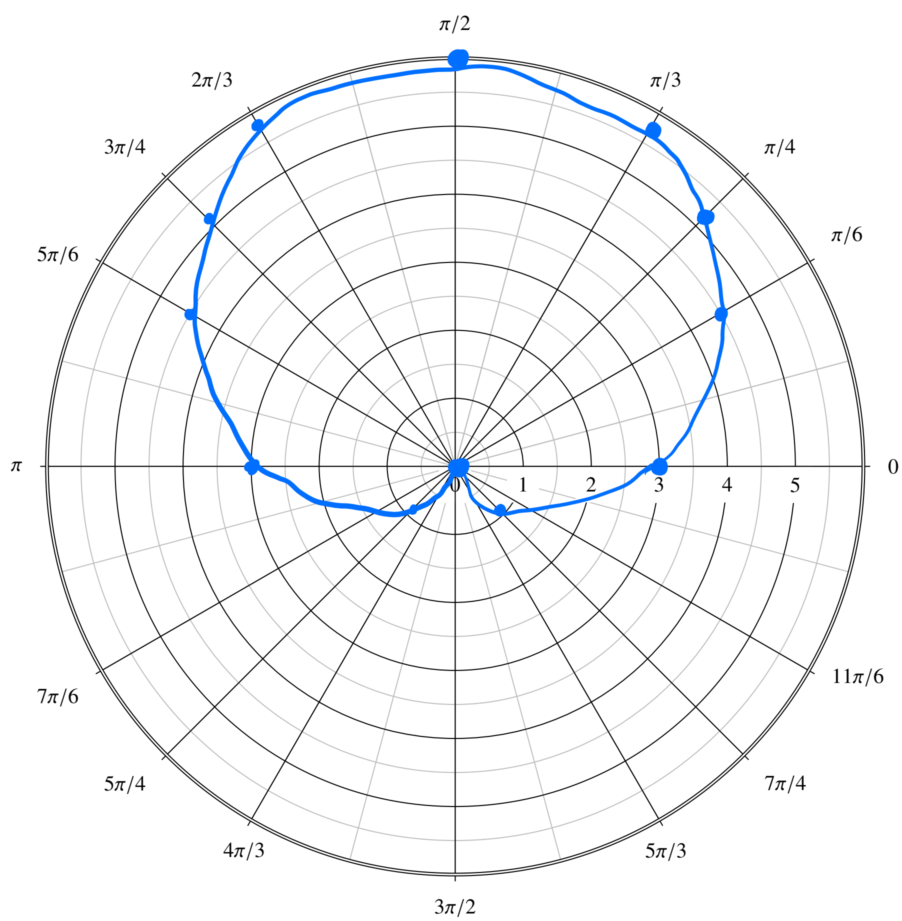
$$x^2 + y^2 = x$$

$$x^2 - x + y^2 = 0$$

$$\left(x - \tfrac{1}{2}\right)^2 + y^2 = \left(\tfrac{1}{2}\right)^2$$

(b)

(c) (*5 pts*) Sketch the polar curve $r = 3 + 3\sin\theta$.



θ	$\sin\theta$	r
0	0	3
$\pi/6$	$1/2$	$3 + \frac{3}{2}$
$\pi/4$	$\sqrt{2}/2$	$3 + \frac{3\sqrt{2}}{2}$
$\pi/3$	$\sqrt{3}/2$	$3 + \frac{3\sqrt{3}}{2}$
$\pi/2$	1	$3 + 3$
$-\frac{\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$3 - \frac{3\sqrt{2}}{2}$
$-\pi/2$	-1	$3 - 3 = 0$

13. (*6 pts*) Evaluate and simplify the indefinite integral $\int e^{2x} \sin x \, dx$.

$u = e^{2x}$, $dv = \sin x \, dx$

$du = 2e^{2x} dx$, $v = -\cos x$

$$= -e^{2x}\cos x + \int 2e^{2x}\cos x \, dx$$

$u = 2e^{2x}$ $dv = \cos x \, dx$

$du = 4e^{2x} dx$ $v = \sin x$

$$= -e^{2x}\cos x + 2e^{2x}\sin x - \int 4e^{2x}\sin x \, dx$$

$$\rightarrow 5\int e^{2x}\sin x \, dx = e^{2x}(2\sin x - \cos x) + C$$

$$\boxed{\int e^{2x}\sin x \, dx = \frac{e^{2x}}{5}(2\sin x - \cos x) + C}$$

Extra Credit A (easier). (*2 pts*) If you compute the millionth partial sum of the infinite series in problem **7 (a)**, about how many digits would you expect to be accurate?

$\sqrt{2} \approx 1.414$

$$b_n = \frac{1}{\sqrt{2n+1}}$$

$$\text{error} < b_{1000001} = \frac{1}{\sqrt{2000003}} \approx \frac{1}{1000 \cdot \sqrt{2}} \approx \frac{1}{1414}$$

$\rightarrow$ accurate up to 3 digits after decimal.

Extra Credit B (harder). (*2 pts*) Evaluate the integral in problem **3 (c)**.

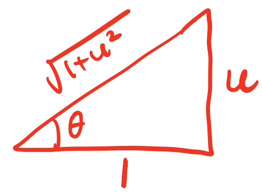
$$\int \cos x \sqrt{1+\sin^2 x}\, dx$$

$u = \sin x$
$du = \cos x\, dx$

$$= \int \sqrt{1+u^2}\, du$$

$\tan\theta = u = \sin x$ $\quad 1+u^2 = 1+\tan^2\theta = \sec^2\theta$
$du = \sec^2\theta\, d\theta$

$$= \int \sec\theta \cdot \sec^2\theta\, d\theta$$



$$= \int \sec^3\theta\, d\theta$$

$\sec\theta = \frac{1}{\cos\theta}$
$= \sqrt{1+u^2}$
$= \sqrt{1+\sin^2 x}$

$$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{1}{2}\sqrt{1+\sin^2 x}\cdot \sin x + \frac{1}{2}\ln\left|\sqrt{1+\sin^2 x} + \sin x\right| + C$$

Integral in 3(c): $\displaystyle\int_0^{\pi/2} 2\pi \cos x \sqrt{1+\sin^2 x}\, dx$

$$= 2\pi\left[\frac{1}{2}\sqrt{1+\sin^2 x}\cdot \sin x + \frac{1}{2}\ln\left|\sqrt{1+\sin^2 x} + \sin x\right|\right]_0^{\pi/2}$$

$$= 2\pi\left[\left(\frac{1}{2}\sqrt{2}\cdot 1 + \frac{1}{2}\ln|\sqrt{2}+1|\right) - \left(\underbrace{\frac{1}{2}\sqrt{1}\cdot 0}_{0} + \underbrace{\frac{1}{2}\ln|\sqrt{1}+0|}_{\ln 1 = 0}\right)\right]$$

$$= 2\pi\cdot\frac{1}{2}\left(\sqrt{2} + \ln(\sqrt{2}+1)\right) = \boxed{\pi\left(\sqrt{2} + \ln(\sqrt{2}+1)\right)}$$

You may find the following **trigonometric formulas** useful. Other formulas, not listed here, should be in your memory, or you can derive them from the ones here.

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$$

$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$$

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$$