Math F252 Midterm I Spring 2026

Name: SOLUTIONS

Rules:

You have 90 minutes to complete this midterm.

Partial credit will be awarded, but you must show your work.

Notes, books, calculators, and internet access are not allowed.

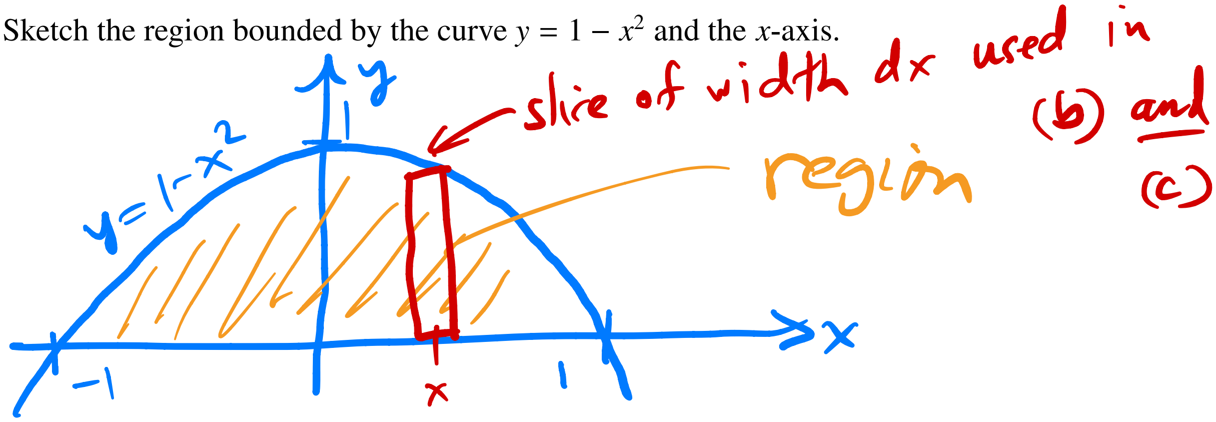
Circle or box your FINAL ANSWER to each question where appropriate.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	16	
2	10	
3	10	
4	8	
5	6	
6	28	
7	10	
8	12	
Extra Credit	3	
Total	100	

1. (a) 2 pts Sketch the region bounded by the curve $y = 1 - x^2$ and the x-axis.



(b) 5 pts Use an integral to calculate the volume of the solid found by rotating the region in part **(a)** around the x-axis.

$\leftarrow$ discs

$$V = \int_{-1}^{1} \pi (1-x^2)^2 \, dx$$

$$= 2\pi \int_0^1 1 - 2x^2 + x^4 \, dx = 2\pi \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_0^1$$

$$= 2\pi \left(1 - \frac{2}{3} + \frac{1}{5}\right) = 2\pi \left(\frac{8}{15}\right) = \frac{16\pi}{15}$$

(c) 5 pts Use an integral to calculate the volume of the solid obtained by rotating the region in part **(a)** around the y-axis. Use the **shell method**.

$$V = \int_0^1 2\pi x (1-x^2) \, dx = 2\pi \int_0^1 x - x^3 \, dx$$

$$= 2\pi \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = 2\pi \left(\frac{1}{2} - \frac{1}{4} \right)$$

$$= \frac{\pi}{2}$$

(d) 4 pts Set up, but **do not evaluate**, an integral for the length of the upper curve of the region sketched in part **(a)**.

$$L = \int_{-1}^{1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$= \int_{-1}^{1} \sqrt{1 + 4x^2} \, dx$$

2. 10 pts A 1-meter long ski pole has linear density $\rho(x) = 1 - \dfrac{1}{x+3}$ grams per centimeter, where one end of the pole is at $x = 0$. Determine the mass of the pole. Include units.

$$m = \int \rho(x)\,dx = \int_0^{100} 1 - \frac{1}{x+3}\,dx$$

$$= \Big[x - \ln|x+3|\Big]_0^{100}$$

$$= (100 - \ln(103)) - (0 - \ln(3))$$

$$= 100 + \ln\left(\frac{3}{103}\right) \text{ grams}$$

3. 10 pts A 10 centimeter spring requires 20 J of work to stretch the spring to a length of 12 centimeters. How much work is required to compress the spring to a length of 5 centimeters? Include units.

$$20 = \int_0^2 kx\,dx = k\cdot\left[\frac{x^2}{2}\right]_0^2 = k\cdot 2$$

$$\therefore \quad k = 10 \text{ N/m}, \quad F = -kx$$

$$W = \int_{-5}^{0} (-kx)\,dx = -\int_{-5}^{0} 10x\,dx$$

$$= -\frac{10}{2}\left[x^2\right]_{-5}^{0} = -5(0-25) = 125 \text{ J}$$

4. 8 pts Evaluate the definite integral, and simplify your answer. (integration-by-parts)

$$\int_0^2 x\,e^{2x}\,dx = \left[x\cdot\frac{e^{2x}}{2}\right]_0^2 - \frac{1}{2}\int_0^2 e^{2x}\,dx$$

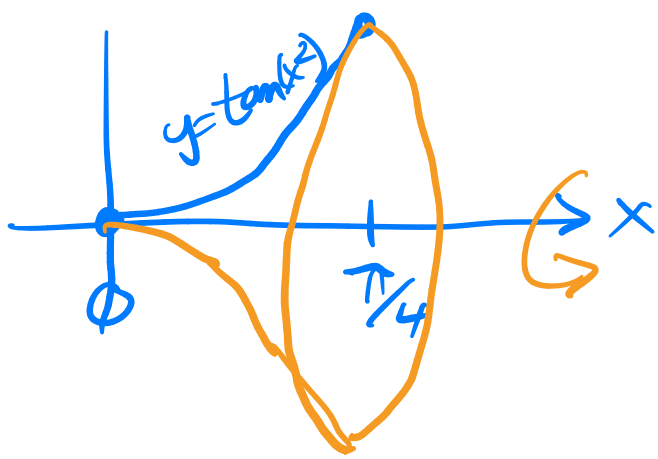
$$\left[\begin{array}{ll} u=x & v=e^{2x}/2 \\ du=dx & dv=e^{2x}dx \end{array}\right]$$

$$= (e^4 - 0) - \frac{1}{2}\left[\frac{e^{2x}}{2}\right]_0^2$$

$$= e^4 - \frac{1}{4}(e^4-1) = \frac{3}{4}e^4 + \frac{1}{4}$$

$$= \frac{3e^4+1}{4}$$

5. 6 pts Set up, **but do not evaluate**, an integral $= f(x)$ which computes the surface area of the surface of revolution generated by revolving $y = \tan(x^2)$, between $x = 0$ to $x = \pi/4$, around the x-axis.



$$f'(x) = \sec^2(x^2)\cdot 2x$$

$$A = \int_0^{\pi/4} 2\pi f(x)\sqrt{1+f'(x)^2}\,dx$$

$$= 2\pi\int_0^{\pi/4} \tan(x^2)\sqrt{1+4x^2\sec^4(x^2)}\,dx$$

use $\sin(ax)\sin(bx) = \cdots$ formula from last page

6. Evaluate the indefinite integrals, and simplify your answers.

(a) 7 pts $\int \sin(3x)\sin(4x)\,dx = \int \frac{1}{2}\cos((3-4)x) - \frac{1}{2}\cos((3+4)x)\,dx$

$\cos(x)$ is even

$$= \frac{1}{2}\int \cos(x)\,dx - \frac{1}{2}\int \cos(7x)\,dx$$

$$= \frac{1}{2}\sin(x) - \frac{1}{14}\sin(7x) + C$$

Double-angle identies

(b) 7 pts $\int \sin^2 t \cos^2 t\,dt = \int \frac{1}{2}(1-\cos(2t))\,\frac{1}{2}(1+\cos(2t))\,dt$

$$= \frac{1}{4}\int 1 - \cos^2(2t)\,dt$$

$$= \frac{1}{4}t - \frac{1}{4}\int \frac{1}{2}(1+\cos(4t))\,dt$$

$$= \frac{1}{4}t - \frac{1}{8}t - \frac{1}{8}\int \cos(4t)\,dt$$

$$= \frac{1}{8}t - \frac{1}{8}\frac{1}{4}\sin(4t) + C$$

$$= \frac{1}{8}t - \frac{1}{32}\sin(4t) + C$$

integration-by-parts

(c) 7 pts $\int \ln(5x)\,dx =$ $x\ln(5x) - \int x \cdot \frac{1}{x}\,dx$

$$u = \ln(5x) \qquad v = x$$
$$du = \frac{1}{5x}\cdot 5\,dx \qquad dv = dx$$
$$= \frac{1}{x}\,dx$$

$$= x\ln(5x) - \int 1\,dx = x\ln(5x) - x + C$$

(d) 7 pts $\int \frac{2}{x^2 - 3x + 2}\,dx =$ $2\int \frac{1}{(x-2)(x-1)}\,dx$

$$\frac{1}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$$
$$0x + 1 = A(x-1) + B(x-2)$$
$$= (A+B)x + (-A-2B)$$

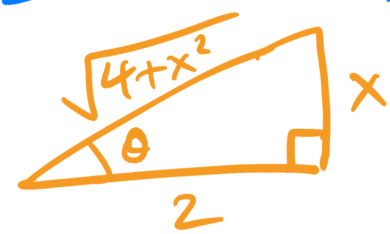
$$\left.\begin{aligned} A+B &= 0 \\ -A-2B &= 1 \end{aligned}\right\} \text{ add: } -B = 1 \quad \therefore B = -1$$
$$\therefore A = 1$$

$$= 2\int \frac{1}{x-2} - \frac{1}{x-1}\,dx = 2\left(\ln|x-2| - \ln|x-1|\right) + C$$

7. 10 pts Use a trigonometric substitution to evaluate the integral. Write your final answer in terms of x, and simplify it.

$$\int \frac{dx}{(4+x^2)^{3/2}} = \int \frac{2\sec^2\theta \, d\theta}{(4+4\tan^2\theta)^{3/2}}$$

$$\left[\begin{array}{l} x = 2\tan\theta \\ dx = 2\sec^2\theta \, d\theta \end{array}\right.$$



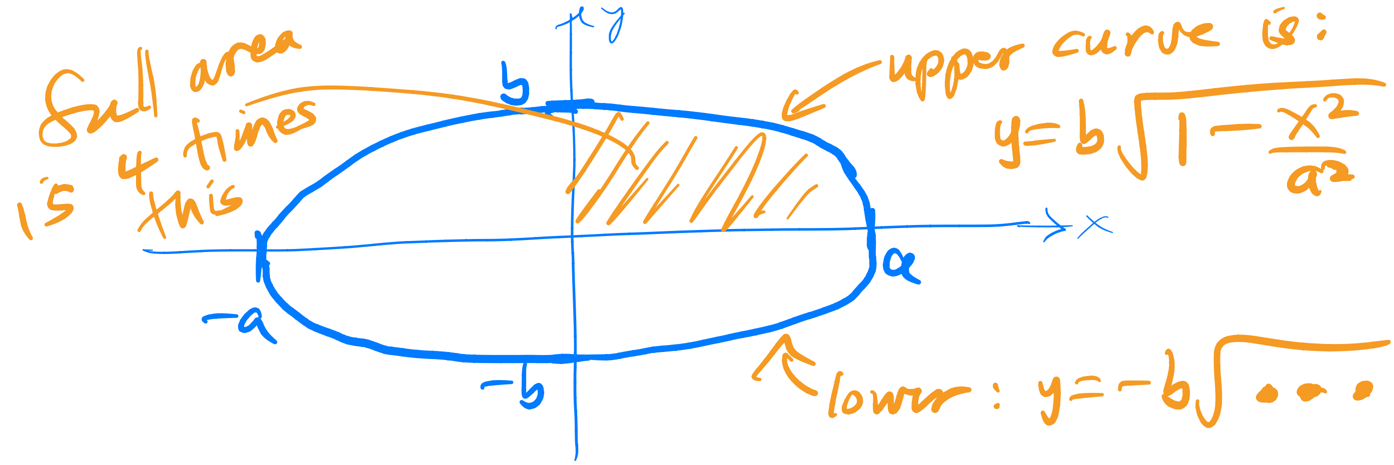
$$= \frac{2}{8}\int \frac{\sec^2\theta}{\sec^3\theta}\, d\theta$$

$$= \frac{1}{4}\int \frac{1}{\sec\theta}\, d\theta$$

$$= \frac{1}{4}\int \cos\theta \, d\theta$$

$$= \frac{1}{4}\sin\theta + C = \frac{1}{4}\frac{x}{\sqrt{4+x^2}} + C$$

8. **(a)** 4 pts For a and b fixed positive numbers, the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an ellipse. Sketch it. (*Hints.* Find the points along the x and y-axes, e.g. by substituting zero for one of the variables.)



(b) 8 pts Determine the area inside the ellipse by computing an integral. (*Hint.* Start by solving for y.)

$$A = \int_{-a}^{a} \left(b\sqrt{1-\frac{x^2}{a^2}}\right) - \left(-b\sqrt{1-\frac{x^2}{a^2}}\right) dx$$

$$= 4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\, dx$$

you may recognize this as area of $\frac{1}{4}$ of unit circle

$$= 4b\int_0^1 \sqrt{1-u^2}\cdot a\, du \qquad \left[\begin{array}{l} u = \frac{x}{a} \\ a\,du = dx \end{array}\right]$$

$$= 4ab \int_0^1 \sqrt{1-u^2}\, du \qquad \left[\begin{array}{l} u = \sin\theta \\ du = \cos\theta\, d\theta \end{array}\right]$$

$$= 4ab\int_0^{\pi/2} \sqrt{1-\sin^2\theta}\, \cos\theta\, d\theta$$

$$= 4ab\int_0^{\pi/2} \cos^2\theta\, d\theta = 2ab\int_0^{\pi/2} 1+\cos(2\theta)\, d\theta$$

$$= 2ab\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\pi/2} = 2ab\left(\frac{\pi}{2}+0-0\right)$$

$$= \pi ab$$

even integrand

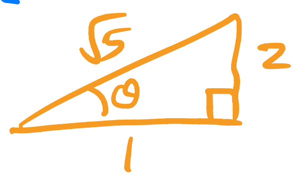
Extra Credit. 3 pts Evaluate, and fully-simplify, the integral you set up in problem **1 (d)**.

$$L = \int_{-1}^{1} \sqrt{1+4x^2}\,dx = 2\int_0^1 \sqrt{1+4x^2}\,dx$$

$$\begin{bmatrix} 2x = \tan\theta \\ 2dx = \sec^2\theta\,d\theta \end{bmatrix} = 2\int_0^{\arctan(2)} \sqrt{1+\tan^2\theta}\cdot\frac{\sec^2\theta\,d\theta}{2}$$

see calculation on other side

$$= \int_0^{\arctan(2)} \sec^3\theta\,d\theta = \left[\frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right)\right]_0^{\arctan(2)}$$



$$= \frac{1}{2}\left(\left(\sqrt{5}\cdot 2 + \ln|\sqrt{5}+2|\right) - \left(0 + \cancelto{0}{\ln 1}\right)\right)$$

$$= \sqrt{5} + \frac{1}{2}\ln\left(2+\sqrt{5}\right)$$

You may find the following **trigonometric formulas** useful. Other formulas, not listed here, should be in your memory, or you can derive them from the ones here.

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \qquad \sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \qquad \sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$$

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$$

E.C.

integration-by-parts

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$$I = \int \sec^3\theta \, d\theta = \sec\theta \tan\theta - \int \sec\theta \tan^2\theta \, d\theta$$

$$\left[\begin{array}{ll} u = \sec\theta & v = \tan\theta \\ du = \sec\theta \tan\theta \, d\theta & dv = \sec^2\theta \, d\theta \end{array}\right]$$

$$= \sec\theta \tan\theta - \int \sec\theta (\sec^2\theta - 1) \, d\theta$$

$$= \sec\theta \tan\theta - \underbrace{\int \sec^3\theta \, d\theta}_{= I} + \int \sec\theta \, d\theta$$

memory

so:

$$2I = \sec\theta \tan\theta + \ln|\sec\theta + \tan\theta|$$

$$I = \frac{1}{2}\left(\sec\theta \tan\theta + \ln|\sec\theta + \tan\theta|\right) + C$$