Math F252 Midterm II Spring 2026

Name: Solutions

Rules:

You have 90 minutes to complete this midterm.

Partial credit will be awarded, but you must show your work.

Calculators are not allowed.

Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question, or use the box provided.

Turn off anything that might go beep during the exam.

You may bring one page of hand-written notes (8.5×11in paper, single side).

Good luck!

Problem	Possible	Score
1	16	
2	18	
3	12	
4	12	
5	12	
6	6	
7	6	
8	18	
Extra Credit	*3*	
Total	100	

1. Compute and simplify the improper integrals, or show they diverge. Use correct limit notation.

(a) (8 pts) $\displaystyle\int_0^\infty xe^{-x}dx =$

$u = x$, $dv = e^{-x}dx$

$du = dx$ $v = -e^{-x}$

$$\int xe^{-x}dx = -xe^{-x} - \int -e^{-x}dx$$

$$= -xe^{-x} - e^{-x} + C = -(x+1)e^{-x} + C$$

$$\int_0^\infty xe^{-x}dx = \lim_{t\to\infty}\left[-(x+1)e^{-x}\right]_0^t = \lim_{t\to\infty}\left[\left(-\frac{(t+1)}{e^t}\right) - \left(-(0+1)e^0\right)\right]$$

$$= 0 + 1$$

$$\boxed{= 1}$$

(b) (8 pts) $\displaystyle\int_1^2 \frac{dx}{\sqrt{2-x}} = \lim_{t\to 2^-}\int_1^t \frac{dx}{\sqrt{2-x}} = \lim_{t\to 2^-}\left[-2\sqrt{2-x}\right]_1^t$

$$= \lim_{t\to 2^-}\left[\left(-2\sqrt{2-t}\right) - \left(-2\sqrt{2-1}\right)\right]$$

$$= 0 + 2$$

$$\boxed{= 2}$$

2. Do the following series converge or diverge? Show your work, including naming any test you use.

(a) (*6 pts*) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

ratio test: $\displaystyle\lim_{n\to\infty} \left| \frac{\frac{(n+1)!^2}{(2n+2)!}}{\frac{(n!)^2}{(2n)!}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{(2n+1)(2n+2)} \right|$

$= \frac{1}{4} < 1$

$\rightarrow$ $\boxed{\text{converges}}$

(b) (*6 pts*) $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^{2n}}{n^n}$

root test: $\displaystyle\lim_{n\to\infty} \sqrt[n]{\frac{(\ln n)^{2n}}{n^n}}$

$\displaystyle = \lim_{n\to\infty} \frac{(\ln n)^2}{n} = 0 < 1$

$\rightarrow$ $\boxed{\text{converges}}$

(c) (*6 pts*) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

$b_n = \frac{1}{\ln n}$ is positive, decreasing, and

$\displaystyle\lim_{n\to\infty} \frac{1}{\ln n} = 0$,

so $\boxed{\text{converges}}$ by Alt. Series Test

3. Do the following series converge absolutely, conditionally, or neither(diverge)? Show your work, identify any test(s) you use, and circle one answer.

(a) (6 *pts*) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$

Check abs. convergence: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

Ratio test: $\displaystyle\lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \right| = \lim_{n\to\infty} \frac{1}{n+1} = 0$

So the series converges absolutely

CONVERGES ABSOLUTELY | CONVERGES CONDITIONALLY | DIVERGES

(a) (6 *pts*) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n} \ln n}{n}$

Check abs. conv.: $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

Comparison test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges:

$\frac{\ln n}{n} > \frac{1}{n}$ for all $n \geq 3$,

so $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges

Since $\frac{\ln n}{n}$ is positive for $n > 1$, decreasing (since $\frac{d}{dx} \frac{\ln x}{x} = \frac{1-\ln x}{x^2} < 0$ for $x > e$)

and $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n} = 0$,

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$ converges by the Alt Series Test

CONVERGES ABSOLUTELY | CONVERGES CONDITIONALLY | DIVERGES

4. Find the **radius** and **interval** of convergence of the following power series. Show your work, identify any test(s) you use, and write your answers in the provided boxes.

(a) (*6 pts*) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 x^n}{2^n}$

Ratio test: $\displaystyle\lim_{n\to\infty} \left| \frac{\frac{(n+1)^2 x^{n+1}}{2^{n+1}}}{\frac{n^2 x^n}{2^n}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2 \cdot x}{n^2 \cdot 2} \right| = \left|\frac{x}{2}\right|$

Converges when $\left|\frac{x}{2}\right| < 1$, i.e. $|x| < 2$.

center = 0

radius = 2

Check endpoints: $\underline{x=2}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 \cdot 2^n}{2^n} = \sum_{n=1}^{\infty} n^2$ diverges

$\underline{x=-2}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 (-2)^n}{2^n} = \sum_{n=1}^{\infty} n^2 (-1)^n$ diverges

$R =$ 2	interval: $(-2, 2)$

(b) (*6 pts*) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$

Ratio test: $\displaystyle\lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{(n+1)^{n+1}} x^{n+1}}{\frac{n!}{n^n} x^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1) \cdot n^n}{(n+1)^{n+1}} \cdot x \right|$

$\displaystyle= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n |x| = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-n} |x|$

$\displaystyle= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n} |x| = e^{-1} \cdot |x|$

$\rightarrow$ converges when $|x| < e$,

radius of convergence $= e$.

$R =$ e	interval: $(-e, e)$

5. Show your work.

(a) (*6 pts*) Find a simplified power series representation for the function $f(x) = \dfrac{x^2}{2-x}$.

$$\frac{x^2}{2-x} = \frac{x^2}{2} \cdot \frac{1}{1-\left(\frac{x}{2}\right)} = \frac{x^2}{2} \cdot \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \frac{x^2}{2} \sum_{n=0}^{\infty} \frac{x^n}{2^n}$$

$$\boxed{= \sum_{n=0}^{\infty} \frac{x^{n+2}}{2^{n+1}}}$$

(b) (*6 pts*) If $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$, find $f'(x)$ and simplify your answer.

$$f'(x) = \sum_{n=1}^{\infty} n \cdot \frac{(-1)^{n-1}}{n} \cdot x^{n-1} = \sum_{n=1}^{\infty} (-x)^{n-1} = \sum_{n=0}^{\infty} (-x)^n$$

$$= \frac{1}{1-(-x)}$$

$$\boxed{= \frac{1}{1+x}}$$

6. (*6 pts*) Find a simplified power series of the function $f(x) = \dfrac{4}{(x-3)(x+1)}$.

$$\frac{4}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} \rightarrow 4 = A(x+1) + B(x-3)$$
$$= (A+B)x + (A-3B)$$

$$\rightarrow \begin{matrix} A+B=0 \\ A-3B=4 \end{matrix} \rightarrow 4 = A-3(-A) = 4A$$
$$\rightarrow A=1, \quad B=-1$$

$$= \frac{1}{x-3} - \frac{1}{x+1} = \left(-\frac{1}{3}\right)\frac{1}{1-\frac{x}{3}} - \frac{1}{1-(-x)}$$

$$= \left(-\frac{1}{3}\right)\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n - \sum_{n=0}^{\infty}(-x)^n$$

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{3^{n+1}}\right)x^n - \sum_{n=0}^{\infty}(-1)^n x^n$$

$$\boxed{= \sum_{n=0}^{\infty}\left(-\frac{1}{3^{n+1}} - (-1)^n\right)x^n}$$

7. (*6 pts*) Evaluate $\displaystyle\sum_{n=1}^{\infty}\frac{n}{2^n}$ by using the derivative of $f(x) = \displaystyle\sum_{n=0}^{\infty} x^n$.

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$f'(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$

$$\frac{1}{2}\cdot f'\left(\frac{1}{2}\right) = \frac{1}{2}\cdot\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^{n-1} = \sum_{n=1}^{\infty}\frac{n}{2^n}$$

$$= \frac{1}{2}\left(\frac{1}{(1-\frac{1}{2})^2}\right) = \frac{1}{2}\left(\frac{1}{(\frac{1}{2})^2}\right) = \frac{1}{2}\left(\frac{1}{\frac{1}{4}}\right) = \frac{1}{2}\cdot 4 \boxed{= 2}$$

8. **(a)** (*6 pts*) Find the Taylor polynomial of degree two approximating $f(x) = e^x$ centered at $a = 0$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (\text{MacLaurin Series})$$

degree 2: $1 + \frac{x^1}{1} + \frac{x^2}{2} \boxed{= 1 + x + \frac{x^2}{2}}$

(b) (*6 pts*) Find the Taylor series for $f(x) = e^x$ centered at $a = 2$.

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n$$

$$\boxed{= \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n}$$

$f^{(n)}(x) = e^x$

$f^{(n)}(2) = e^2$

$\sin(0) = 0$

$\cos(0) = 1$

(c) (*6 pts*) Find the MacLaurin series for $f(x) = \sin(2x)$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$\boxed{= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^{2n+1}}{(2n+1)!} x^{2n+1}}$$

$$\left(= 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \cdots \right)$$

$f(x) = \sin(2x)$ $\qquad f(0) = 0$

$f^{(1)}(x) = 2\cos(2x)$ $\qquad f^{(1)}(0) = 2$

$f^{(2)}(x) = -4\sin(2x)$ $\qquad f^{(2)}(0) = 0$

$f^{(3)}(x) = -8\cos(2x)$ $\qquad f^{(3)}(0) = -8$

$f^{(4)}(x) = 16\sin(2x)$ $\qquad f^{(4)}(0) = 0$

$\vdots$

For $n \geq 0$: $\quad f^{(2n)}(0) = 0$

$f^{(2n+1)}(0) = (-1)^{n+1} \cdot 2^{2n+1}$

Extra Credit. (*3 pts*) Find a power series representation of $f(x) = \arctan(x)$. Show your work. (**Hint:** Using Taylor's formula for the coefficients is **not** required.)

$$\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\arctan(x) = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} + C$$

Since $\arctan(0) = 0$, $\quad C = 0$.

$$\to \arctan(x) = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}}$$

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