Math 252: Quiz 11 **8 December, 2022**

Name: SOLUTIONS

/ 25

30 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form. 25 points possible.

1. [8 points] Consider the parametric curve

$$x(t) = 5\cos t, \qquad y(t) = \sin t$$

a. Determine the slope and the equation of the tangent line at $t = \pi/2$.

$$m = \left.\frac{dy}{dx}\right|_{t=\pi/2} = \left.\frac{dy/dt}{dx/dt}\right|_{t=\pi/2}$$

$$= \left.\frac{\cos t}{-5\sin t}\right|_{t=\pi/2} = \frac{0}{-5} = 0$$

$$x(\pi/2) = 0,\ y(\pi/2) = 1 \Rightarrow y - 1 = 0(x - 0)$$

(slope) = $\boxed{0}$ equation: $\boxed{y = 1}$

b. Eliminate the parameter t to write the curve in rectangular form.

$$\frac{x}{5} = \cos t, \quad y = \sin t$$

(use $\cos^2 t + \sin^2 t = 1$)

$$\frac{x^2}{5^2} + y^2 = 1$$

2. [5 points] Fully set up, but do not evaluate, an integral for the length of the spiral curve $x(t) = t\cos t$, $y(t) = t\sin t$ from $t = 0$ to $t = 2\pi$.

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2}\, dt$$

3. [5 points] Find $\dfrac{d^2y}{dx^2}$:

$$x = t^2 - t, \qquad y = t + e^t$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{1+e^t}{2t-1}\right)}{2t-1}$$

$$= \frac{\frac{e^t(2t-1) - (1+e^t)2}{(2t-1)^3}}{2t-1}$$

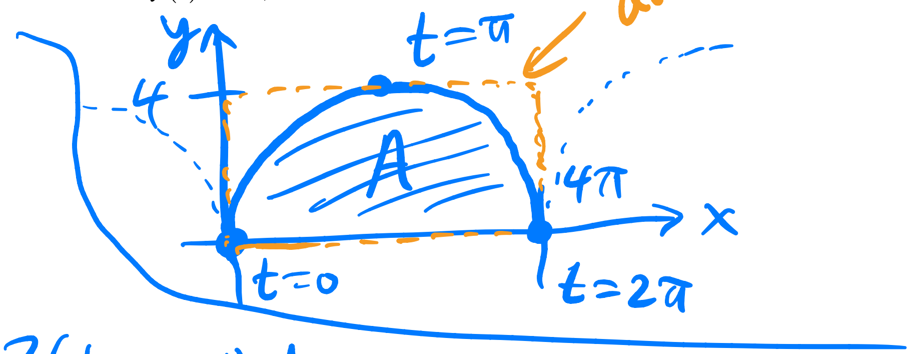
$$= \frac{e^t(2t-3) - 2}{(2t-1)^3}$$

4. [7 points] Find the area under one hump of the cycloid

$$x(t) = 2(t - \sin t), \quad y(t) = 2(1 - \cos t)$$

(**Hint.** One hump goes from $t = 0$ to the next t where $y(t) = 0$.)

rectangle has area 16π



$$A = \int y \frac{dx}{dt}\, dt$$

$$= \int_0^{2\pi} 2(1-\cos t) \cdot 2(1-\cos t)\, dt$$

$$= 4 \int_0^{2\pi} 1 - 2\cos t + \cos^2 t\, dt$$

$$= 4 \int_0^{2\pi} 1 - 2\cos t + \frac{1}{2} + \frac{1}{2}\cos(2t)\, dt$$

$$= 4 \left[\frac{3}{2} t - 2\sin t + \frac{1}{4}\sin(2t) \right]_0^{2\pi}$$

$$= 4\left(\left(\frac{3}{2} \cdot 2\pi - 0 + 0\right) - (0 - 0 + 0)\right)$$

$$= 12\pi$$

3 valid methods shown

EC. [1 points] (**Extra Credit**) Eliminate the parameter to write the curve $x(t) = \sin(2t)$, $y(t) = 2\sin t$ in rectangular form.

① $x = 2\sin t\cos t = y\cos t = y\pm\sqrt{1-\sin^2 t}$

$x = \pm y\sqrt{1-\left(\frac{y}{2}\right)^2}$

② $x = 2\sin t\cos t = y\cos t \quad \therefore \frac{x}{y} = \cos t, \ \frac{y}{2} = \sin t$

$\therefore \left(\frac{x}{y}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$

③ $t = \frac{1}{2}\arcsin(x) \quad \therefore \quad y = 2\sin t = 2\sin\left(\frac{\arcsin x}{2}\right)$

EXTRA SPACE FOR ANSWERS