

Name: Solutions

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30 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form. 25 points possible.

1. (8 points) Use the (direct) comparison test to determine whether the series converge or diverge. You must include a detailed implementation of the test. You must (i) identify the comparison series, (ii) indicate whether the comparing series converges or diverges and why, (iii) demonstrate a correct inequality, and (iv) draw a conclusion.

(a) $\sum_{n=1}^{\infty} \frac{1}{3n-1}$

Compare to $\frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{3n}$, harmonic series which diverges.

Since $3n-1 < n$, we know $\frac{1}{3n-1} > \frac{1}{3n}$.

Since $\frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \frac{1}{3n-1}$ diverges.

(b) $\sum_{n=1}^{\infty} \frac{\cos^2(n)}{2^n}$

Compare to $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$, a convergent geometric series.

Since $0 \leq \cos^2(n) \leq 1$, $\frac{\cos^2(n)}{2^n} \leq \frac{1}{2^n}$.

Since $\sum \left(\frac{1}{2}\right)^n$ converges, $\sum \frac{\cos^2(n)}{2^n}$ converges.

2. (9 points) Use the limit comparison test to determine whether the series converge or diverge. You must include a detailed implementation of the test. You must (i) identify the comparison series, (ii) indicate whether the comparing series converges or diverges and why, (iii) evaluate an appropriate limit, and (iv) draw a conclusion.

(a) $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^2 - \sqrt{n}}$

Compare to $\sum \frac{1}{n^{3/2}}$, a convergent p-series ($p = \frac{3}{2} > 1$)

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n^2 - \sqrt{n}} \cdot \frac{n^{3/2}}{1} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 - \sqrt{n}} = 1$$

Since $\sum \frac{1}{n^{3/2}}$ converges, $\sum \frac{\sqrt{n}}{n^2 - \sqrt{n}}$ also converges

(b) $\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^3}$

Compare to $\sum \frac{1}{n}$, the harmonic series which diverges.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1}{(\ln(n))^3} \cdot \frac{n}{1} = \lim_{n \rightarrow \infty} \frac{n}{(\ln(n))^3} \stackrel{(A)}{=} \lim_{n \rightarrow \infty} \frac{1}{3(\ln(n))^2 \cdot \frac{1}{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{n}{3(\ln(n))^2} \stackrel{(A)}{=} \lim_{n \rightarrow \infty} \frac{n}{6 \ln(n)} \stackrel{(A)}{=} \lim_{n \rightarrow \infty} \frac{n}{6} = \infty.$$

Since $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, the series

$$\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^3} \text{ diverges.}$$

3. (8 points) Determine if the series below are convergent or divergent. You must include a justification. You must (i) state the test (or tests) you are applying, (ii) apply the test (or tests), and (iii) draw a conclusion.

(a) $\sum_{n=1}^{\infty} (-1)^n \ln\left(1 + \frac{1}{n}\right)$

A.S.T. with $b_n = \ln\left(1 + \frac{1}{n}\right)$

① $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \ln\left(1 + \frac{1}{n}\right) = 0$

② Since $1 + \frac{1}{n+1} < 1 + \frac{1}{n}$, then
 $b_{n+1} = \ln\left(1 + \frac{1}{n+1}\right) < \ln\left(1 + \frac{1}{n}\right) = b_n$

So the series $\sum (-1)^n \ln\left(1 + \frac{1}{n}\right)$ converges

(b) $\sum_{n=2}^{\infty} (-1)^n \frac{100n+50}{n-1}$

Divergence Test

$\lim_{n \rightarrow \infty} \frac{100n+50}{n-1} = 100 \neq 0.$

So $\lim_{n \rightarrow \infty} (-1)^n \left(\frac{100n+50}{n-1}\right) = \text{DNE}.$

So the series diverges.