

Name: _____

/ 25

30 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form. 25 points possible.

1. (5 points) A 1-dimensional rod is 4 meters long (starting at $x = 2$) and has density function $\rho(x) = \frac{5}{x}$ grams per meter. Find its mass. Include units in your final answer.

Solution

The rod runs from $x = 2$ to $x = 6$.

$$m = \int_2^6 \frac{5}{x} dx = 5 \left[\ln |x| \right]_2^6 = 5(\ln 6 - \ln 2) = 5 \ln 3.$$

$$m = 5 \ln(3) \text{ grams}$$

2. (8 points) A 2 meter spring requires 40 joules to stretch from 2 meters to 2.2 meters. How much work would it take to stretch the spring from 2 meters to 2.1 meters? Include units in your final answer.

Solution

Measure x from the natural length of 2 meters, so $F(x) = kx$. Stretching to 2.2 meters is $x = 0.2$:

$$40 = \int_0^{0.2} kx dx = k \frac{(0.2)^2}{2} = 0.02k \quad \implies \quad k = 2000 \text{ N/m.}$$

Stretching to 2.1 meters is $x = 0.1$:

$$W = \int_0^{0.1} 2000x dx = 2000 \frac{(0.1)^2}{2} = 10.$$

$$W = 10 \text{ joules}$$

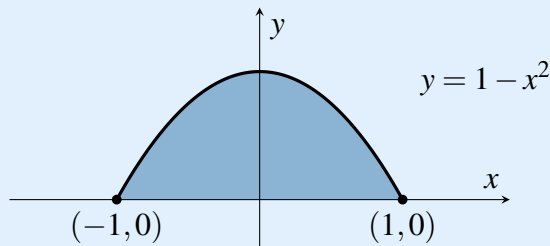
3. Consider the region bounded by the curves $y = 1 - x^2$ and $y = 0$. You may find the following helpful:

$$m = \rho \int_a^b f(x) dx, M_x = \rho \int_a^b \frac{(f(x))^2}{2} dx, M_y = \rho \int_a^b x f(x) dx$$

- (a) (8 points) Find the y coordinate of the centroid (center of mass).

Solution

The curves meet where $1 - x^2 = 0$, so $a = -1$ and $b = 1$, with $f(x) = 1 - x^2$.



$$m = \rho \int_{-1}^1 (1 - x^2) dx = \rho \left[x - \frac{x^3}{3} \right]_{-1}^1 = \frac{4\rho}{3},$$

$$\begin{aligned} M_x &= \rho \int_{-1}^1 \frac{(1 - x^2)^2}{2} dx = \frac{\rho}{2} \int_{-1}^1 (1 - 2x^2 + x^4) dx \\ &= \frac{\rho}{2} \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_{-1}^1 = \frac{\rho}{2} \cdot \frac{16}{15} = \frac{8\rho}{15}. \end{aligned}$$

$$\bar{y} = \frac{M_x}{m} = \frac{8\rho/15}{4\rho/3} = \frac{8}{15} \cdot \frac{3}{4} \quad \boxed{\bar{y} = \frac{2}{5}}$$

- (b) (BONUS 2 points) Find the x coordinate of the centroid **without using calculus**. Briefly justify your answer.

Solution

Replacing x by $-x$ leaves both $y = 1 - x^2$ and $y = 0$ unchanged, so the region is symmetric about the y -axis and its centroid must lie on that axis.

$$\boxed{\bar{x} = 0}$$

4. (4 points) Compute the derivative of $f(x) = 5^{3x}$.

Solution

$$f'(x) = 5^{3x} \ln(5) \cdot \frac{d}{dx}(3x)$$

$$\boxed{f'(x) = 3 \ln(5) 5^{3x}}$$