Name: _____ / 25

30 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form. 25 points possible.

1. Consider the sequence defined by $a_1 = 2$ and $a_n = -\frac{1}{2}a_{n-1}$.

   (a) (2 points) Write out the first 5 terms of the sequence. Simplify each term.

   Solution

   $$\boxed{a_1 = 2, \quad a_2 = -1, \quad a_3 = \tfrac{1}{2}, \quad a_4 = -\tfrac{1}{4}, \quad a_5 = \tfrac{1}{8}}$$

   (b) (4 points) Find a formula (closed form) for the nth term a_n of the sequence.

   Solution

   $$\boxed{a_n = 2\left(-\frac{1}{2}\right)^{n-1}}$$

   (c) (3 points) Is the sequence bounded? Justify your answer. If it is bounded, clearly state the bounds.

   Solution

   Yes. $|a_n| = 2\left(\frac{1}{2}\right)^{n-1}$ shrinks as n grows, so the largest positive term is $a_1 = 2$ and the most negative term is $a_2 = -1$: $\boxed{-1 \le a_n \le 2 \text{ for all } n}$

   (d) (3 points) Is the sequence monotone? Justify your answer.

   Solution

   The signs of a_n alternate for every n. $\boxed{\text{Not monotone}}$

   (e) (3 points) What (if anything) can you say about the sequence based on your answers to parts (c) and (d)?

   Solution

   The Monotone Convergence Theorem needs a sequence that is both bounded and monotone. This one is not monotone, so the theorem does not apply.

   $$\boxed{\text{Nothing: (c) and (d) do not determine whether } \{a_n\} \text{ converges.}}$$

2. Compute the following integrals or show that they diverge. Show all work and use proper notation.

(a) (5 points) $\displaystyle\int_1^\infty \frac{\ln(x)}{x}\,dx$

Solution

$$\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x}\,dx.$$

With $u=\ln x$, $du=\frac{dx}{x}$: $\displaystyle\int \frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2}+C$, so

$$\lim_{t\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^t = \lim_{t\to\infty}\frac{(\ln t)^2}{2} = \infty.$$

$$\boxed{\int_1^\infty \frac{\ln x}{x}\,dx \text{ diverges}}$$

(b) (5 points) $\displaystyle\int_0^9 \frac{dx}{\sqrt{x}}$

Solution

$$\int_0^9 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+}\int_t^9 x^{-1/2}\,dx = \lim_{t\to 0^+}\left[2\sqrt{x}\right]_t^9 = \lim_{t\to 0^+}\left(6-2\sqrt{t}\right) = \boxed{6}$$

(c) BONUS (2 points) Compute the following integral. $\displaystyle\int_0^3 \frac{1}{x-1}\,dx$

Solution

The integrand is undefined at $x=1$, inside $[0,3]$:

$$\int_0^3 \frac{dx}{x-1} = \lim_{s\to 1^-}\int_0^s \frac{dx}{x-1} + \lim_{t\to 1^+}\int_t^3 \frac{dx}{x-1}.$$

$$\lim_{t\to 1^+}\int_t^3 \frac{dx}{x-1} = \lim_{t\to 1^+}\left[\ln|x-1|\right]_t^3 = \lim_{t\to 1^+}\left(\ln 2 - \ln(t-1)\right) = \infty.$$

Since one piece diverges,

$$\boxed{\int_0^3 \frac{dx}{x-1} \text{ diverges}}$$