Name: _____ / 25

30 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form. 25 points possible.

1. Consider the following series $\sum_{k=1}^{\infty}\left(\sqrt{k}-\sqrt{k+1}\right)$.

   (a) (2 points) Write out the first four partial sums.

   Solution

   $S_1 = 1-\sqrt{2}$

   $S_2 = \left(1-\sqrt{2}\right)+\left(\sqrt{2}-\sqrt{3}\right) = 1-\sqrt{3}$

   $S_3 = \left(1-\sqrt{2}\right)+\left(\sqrt{2}-\sqrt{3}\right)+\left(\sqrt{3}-2\right) = 1-2 = -1$

   $S_4 = \left(1-\sqrt{2}\right)+\left(\sqrt{2}-\sqrt{3}\right)+\left(\sqrt{3}-2\right)+\left(2-\sqrt{5}\right) = 1-\sqrt{5}$

   (b) (3 points) Write a formula for the nth partial sum.

   Solution

   This is a telescoping series.

   $$S_n = \left(\sqrt{1}-\sqrt{2}\right)+\left(\sqrt{2}-\sqrt{3}\right)+\cdots+\left(\sqrt{n}-\sqrt{n+1}\right) = \boxed{1-\sqrt{n+1}}$$

   (c) (5 points) Determine whether the series converges or diverges. If the series converges, find its value.

   Solution

   $$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\sqrt{n+1}\right) = -\infty$$

   The partial sums diverge, so the series $\boxed{\text{diverges}}$.

2. (5 points) Determine whether the following series converges or diverges. If it converges, find its value.

   $$\sum_{n=1}^{\infty} 500\left(\frac{1}{5^n}\right)$$

   Solution

   Shifting the index to start at $n=0$,

   $$\sum_{n=1}^{\infty} 500\left(\frac{1}{5^n}\right) = \sum_{n=0}^{\infty} 500\left(\frac{1}{5^{n+1}}\right) = \sum_{n=0}^{\infty} \frac{500}{5}\left(\frac{1}{5}\right)^n = \sum_{n=0}^{\infty} 100\left(\frac{1}{5}\right)^n.$$

   This is a geometric series with $a=100$ and $r=\frac{1}{5}$. Since $|r| = \frac{1}{5} < 1$, it converges, and

   $$\sum_{n=0}^{\infty} 100\left(\frac{1}{5}\right)^n = \frac{a}{1-r} = \frac{100}{1-\frac{1}{5}} = \frac{100}{\frac{4}{5}} = \boxed{125}$$

3. For each of the following series, determine whether the series converges or diverges.

(a) (4 points) $\displaystyle\sum_{n=5}^{\infty} 2^{(1/n)}$

Solution

$$\lim_{n\to\infty} 2^{1/n} = 2^0 = 1 \neq 0,$$

so by the divergence test the series $\boxed{\text{diverges}}$.

(b) (4 points) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{1+n^2}$

Solution

Use the integral test with $f(x) = \dfrac{x}{1+x^2}$, so that $f(n) = \dfrac{n}{1+n^2}$. On $[1,\infty)$, f is positive and continuous, and it is decreasing since $f'(x) = \dfrac{1-x^2}{(1+x^2)^2} \leq 0$ for $x \geq 1$.

$$\begin{aligned}\int_1^{\infty} \frac{x}{1+x^2}\,dx &= \lim_{t\to\infty} \int_1^t \frac{x}{1+x^2}\,dx = \lim_{t\to\infty} \left[\tfrac{1}{2}\ln\left(1+x^2\right)\right]_1^t \\ &= \lim_{t\to\infty} \left(\tfrac{1}{2}\ln\left(1+t^2\right) - \tfrac{1}{2}\ln 2\right) = \infty\end{aligned}$$

The integral diverges, so by the integral test, the series $\boxed{\text{diverges}}$.

(c) (2 points) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$

Solution

This is a p-series with $p = 4 > 1$, so it $\boxed{\text{converges}}$.