Name: _____ / 25

30 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form. 25 points possible.

1. For each of the following series, determine whether the series converges or diverges. Answers without justification will receive no credit.

(a) (5 points) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{2\sqrt{n}+5}$

Solution

Compare with $b_n = \dfrac{1}{\sqrt{n}}$. Both $a_n = \dfrac{1}{2\sqrt{n}+5}$ and b_n are positive for $n \geq 3$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{2\sqrt{n}+5} = \lim_{n\to\infty} \frac{1}{2+\frac{5}{\sqrt{n}}} = \frac{1}{2},$$

which is finite and positive. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2} \leq 1$, so it diverges. By the limit comparison test, the original series $\boxed{\text{diverges}}$.

(b) (5 points) $\displaystyle\sum_{n=1}^{\infty} \sin^2(n) \left(\frac{1}{3}\right)^n$

Solution

Since $0 \leq \sin^2(n) \leq 1$,

$$0 \leq \sin^2(n) \left(\frac{1}{3}\right)^n \leq \left(\frac{1}{3}\right)^n.$$

$\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ is a geometric series with $|r| = \frac{1}{3} < 1$, so it converges. By the comparison test, the original series $\boxed{\text{converges}}$.

2. For each of the following series, determine whether the series converges absolutely, converges conditionally, or diverges. Answers without justification will receive no credit.

(a) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{e^{1/n}}$

Solution

$$\lim_{n\to\infty} \left| \frac{(-1)^n}{e^{1/n}} \right| = \lim_{n\to\infty} \frac{1}{e^{1/n}} = \frac{1}{e^0} = 1,$$

so the terms do not approach 0 (they alternate near ± 1). By the divergence test, the series $\boxed{\text{diverges}}$.

(b) (5 points) $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{2\sqrt{n}+5}$

Solution

Absolute convergence. $\displaystyle\sum_{n=3}^{\infty} \left| \frac{(-1)^{n-1}}{2\sqrt{n}+5} \right| = \sum_{n=3}^{\infty} \frac{1}{2\sqrt{n}+5}$, which diverges by Problem 1(a).

Alternating series test. Let $b_n = \dfrac{1}{2\sqrt{n}+5}$. Then $b_n > 0$; b_n is decreasing, since $2\sqrt{n}+5$ is increasing; and $\lim_{n\to\infty} b_n = 0$. So the series converges.

$\boxed{\text{converges conditionally}}$

(c) (5 points) $\displaystyle\sum_{n=10}^{\infty} \frac{(-1)^{n+1}}{n^2 \ln(n)}$

Solution

For $n \geq 10$, $\ln(n) \geq \ln(10) > 1$, so

$$0 < \left| \frac{(-1)^{n+1}}{n^2 \ln(n)} \right| = \frac{1}{n^2 \ln(n)} \leq \frac{1}{n^2}.$$

$\displaystyle\sum_{n=10}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$, so it converges. By the comparison test, $\displaystyle\sum_{n=10}^{\infty} \frac{1}{n^2 \ln(n)}$ converges, so the series $\boxed{\text{converges absolutely}}$.

BONUS (2 points) Does the series $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\cos\left(\frac{1}{n}\right)\right)$ converge or diverge? Answers without justification will receive no credit.

Solution

$$\lim_{n\to\infty}\left(\frac{1}{n}-\cos\left(\frac{1}{n}\right)\right)=0-\cos(0)=-1\neq 0,$$

so by the divergence test the series $\boxed{\text{diverges}}$.