**Name:** _____ / 25

30 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form. 25 points possible.

1. Determine whether the following series converge or diverge.

   (a) (5 points) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{e}+\frac{1}{n}\right)^n$

   **Solution**

   The terms are positive, so we may use the root test.

   $$\lim_{n\to\infty}\sqrt[n]{\left(\frac{1}{e}+\frac{1}{n}\right)^n}=\lim_{n\to\infty}\left(\frac{1}{e}+\frac{1}{n}\right)=\frac{1}{e}<1,$$

   so by the root test the series $\boxed{\text{converges}}$.

   (b) (5 points) $\displaystyle\sum_{n=100}^{\infty}\frac{(n!)^2}{(3n)!}$

   **Solution**

   The terms are positive, so we may use the ratio test.

   $$\frac{a_{n+1}}{a_n}=\frac{\big((n+1)!\big)^2}{(3n+3)!}\cdot\frac{(3n)!}{(n!)^2}=\frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)}=\frac{n+1}{3(3n+2)(3n+1)}$$

   $$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=0<1,$$

   so by the ratio test the series $\boxed{\text{converges}}$.

2. Consider the power series $\sum_{n=0}^{\infty} \frac{(3x)^n}{2^n}$.

(a) (5 points) Determine the radius of convergence of the power series.

**Solution**

$\sum_{n=0}^{\infty} \frac{(3x)^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{3x}{2}\right)^n$ is a geometric series with $r = \frac{3x}{2}$, so it converges exactly when

$$\left|\frac{3x}{2}\right| < 1 \iff |x| < \frac{2}{3}.$$

$$\boxed{R = \frac{2}{3}}$$

(b) (5 points) Determine the interval of convergence of the power series.

**Solution**

Check the endpoints.

$x = \frac{2}{3}$: $\sum_{n=0}^{\infty} 1^n$; the terms do not approach 0, so it diverges by the divergence test.

$x = -\frac{2}{3}$: $\sum_{n=0}^{\infty} (-1)^n$; the terms do not approach 0, so it diverges by the divergence test.

$$\boxed{\left(-\frac{2}{3}, \frac{2}{3}\right)}$$

3. (5 points) Use the fact that $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ with interval of convergence $(-1,1)$ to find a power series representation for $\dfrac{3}{1+5x}$.

**Solution**

Replace $x$ with $-5x$:

$$\frac{3}{1+5x} = 3 \cdot \frac{1}{1-(-5x)} = 3\sum_{n=0}^{\infty}(-5x)^n = \boxed{\sum_{n=0}^{\infty} 3(-5)^n x^n}$$

valid for $|-5x| < 1$, i.e. $-\frac{1}{5} < x < \frac{1}{5}$.

4. (BONUS) (2 points) What is the center of the power series $\sum_{n=0}^{\infty} \dfrac{3(x+5)^n}{4(n+1)}$? (This is not meant to be a challenging problem.)

**Solution**

$x+5 = x-(-5)$, so the center is $\boxed{-5}$.