Math 252 (Bueler): Quiz 6 **6 March 2024**

Name: SOLUTIONS

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30 minutes. No aids (book, notes, calculator, internet, etc.) are permitted. Show all work and use proper notation for full credit. Put answers in reasonably-simplified form. 25 points possible.

1. [12 points] Compute the following improper integrals, or show that they diverge. *Use appropriate limit notation for improper integrals.*

a. $\int_0^{\infty} x e^{-2x}\, dx = \lim_{b\to\infty} \int_0^b x e^{-2x} dx = \lim_{b\to\infty} \left[x \frac{e^{-2x}}{-2}\right]_0^b - \int_0^b \frac{e^{-2x}}{-2} dx$

$\left[\begin{array}{c|c} u=x & v=\dots \\ du=dx & dv=e^{-2x}dx \end{array}\right]$

$= \lim_{b\to\infty} -be^{-2b} + 0 + \frac{1}{2}\int_0^b e^{-2x} dx$

$= 0 + \frac{1}{2} \lim_{b\to\infty} \left[\frac{e^{-2x}}{-2}\right]_0^b = \frac{1}{4}\lim_{b\to\infty}(e^{-2b} + e^0) = \frac{1}{4}$

b. $\int_{-\infty}^{0} \cos\theta\, d\theta = \lim_{a\to-\infty} \int_a^0 \cos\theta\, d\theta$

$= \lim_{a\to-\infty} [\sin\theta]_0^a = \lim_{a\to-\infty} \sin a$ DNE

(diverges)

c. $\int_1^3 \frac{1}{\sqrt{3-x}} dx = \lim_{b\to 3^-} \int_1^b (3-x)^{-1/2} dx$ ← improper at $x=3$

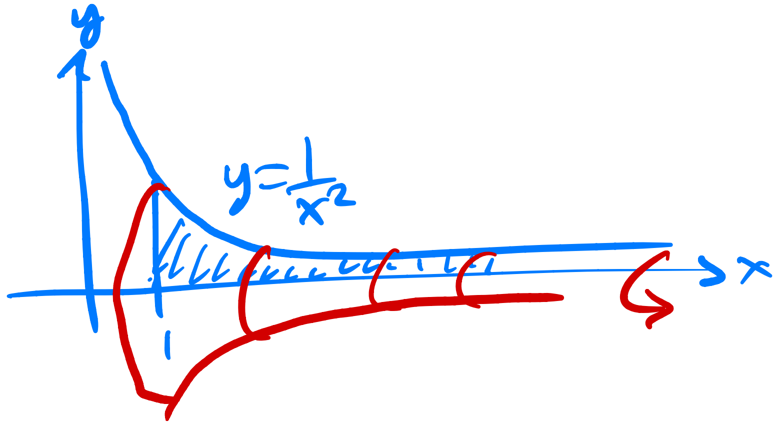
$\left[\begin{array}{c} u=3-x \\ -du=dx \end{array}\right]$

$= \lim_{b\to 3^-} -\int_2^{3-b} u^{-1/2} du = \lim_{b\to 3^-} \int_{3-b}^2 u^{-1/2} du$

$= \lim_{b\to 3^-} \left[2u^{1/2}\right]_{3-b}^2 = \lim_{b\to 3^-} 2\sqrt{2} - \underbrace{2\sqrt{3-b}}_{\lim = 0}$

$= 2\sqrt{2}$

2. [6 points] Sketch the region under the graph $y = \dfrac{1}{x^2}$ on the interval $1 \le x < \infty$. Now find the volume of the solid from rotating this region around the x-axis.



$$V = \int_1^{\infty} \pi \left(\frac{1}{x^2}\right)^2 dx = \lim_{b\to\infty} \int_1^b \pi x^{-4} dx$$

$$= \frac{\pi}{-3} \lim_{b\to\infty} \left[x^{-3}\right]_1^b = -\frac{\pi}{3} \lim_{b\to\infty} \left(b^{-3} - 1\right)$$

$$= -\frac{\pi}{3}(0-1) = \frac{\pi}{3}$$

3. [4 points] Find the general solution of the differential equation $x' = t\sqrt{4+t}$.

$$x(t) = \int t\sqrt{4+t}\,dt = \int (u-4)\sqrt{u}\,du$$

$[u = 4+t,\ du = dt]$

$$= \int u^{3/2} - 4u^{1/2}\,du = \frac{2}{5}u^{5/2} - 4\cdot\frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{5}(4+t)^{5/2} - \frac{8}{3}(4+t)^{3/2} + C$$

4. [3 points] Find the particular solution of the differential equation $y' = 2xy$ which passes through $\left(0, \frac{1}{2}\right)$ given that $y = Ce^{x^2}$ is the general solution.

$x=0,\quad y=\frac{1}{2}$:

$$\frac{1}{2} = Ce^{0^2} = C$$

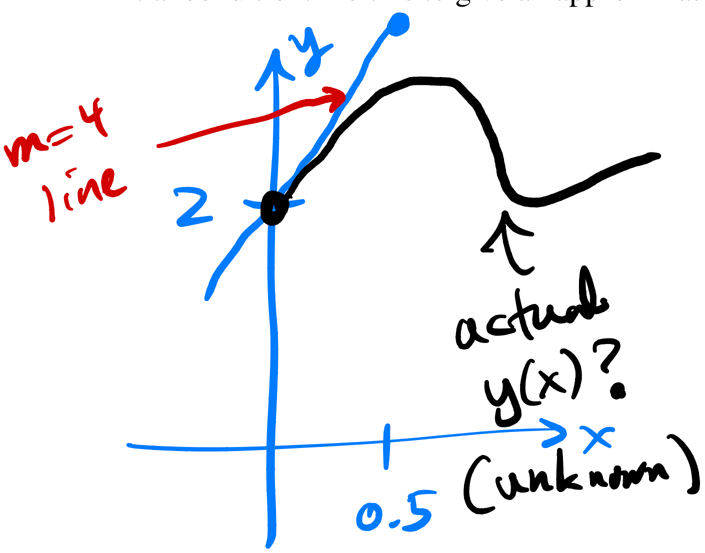
$\therefore$

$$y(x) = \frac{1}{2}e^{x^2}$$

Extra Credit. [1 point] I have no idea how to solve the differential equation

$$y' = \sin(\pi x) + y^2$$

by hand. However, assume the initial condition $y(0) = 2$. Then I can *approximately* compute $y(x)$, at least somewhat beyond $x = 0$, by using the differential equation to create a straight line from the initial condition. Do this to give an approximation to $y(0.5)$.



at $(0,2)$:

$y' = \sin(\pi \cdot 0) + 2^2 = 4$

line:

$y = 2 + 4x$

$y(0.5) \approx 2 + 4(0.5) = 4$

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