Math 252 (Bueler): Quiz 7 **21 March 2024**

Name: SOLUTIONS ______ / 25

30 minutes. No aids (book, notes, calculator, internet, etc.) are permitted. Show all work and use proper notation for full credit. Put answers in reasonably-simplified form. 25 points possible.

1. [3 points] Find a formula for the nth term a_n of the sequence whose first several terms are

$0, 3, 8, 15, 24, 35, 48, 63, 80, 99, \ldots$

$0 = 1 - 1$

$3 = 4 - 1 = 2^2 - 1$

$8 = 9 - 1 = 3^2 - 1$

$15 = 16 - 1 = 4^2 - 1$

$\vdots$

$a_n = n^2 - 1$

$(n \geq 1)$

2. [5 points] Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

a. Use partial fractions and "telescoping" to write a simplified formula for the partial sum S_k.

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$0n + 1 = A(n+1) + Bn$

$= (A+B)n + A$

$A + B = 0 \to \underline{B = -1}$

$\underline{A = 1}$

$$S_k = \sum_{n=1}^{k} \frac{1}{n(n+1)}$$

$$= \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \ldots + \cancel{\frac{1}{k}} - \frac{1}{k+1} = 1 - \frac{1}{k+1}$$

b. Compute the sum of the infinite series.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{k \to \infty} S_k = 1 - 0 = 1$$

3. [6 points] For each sequence, find the limit if it converges, or show that the sequence diverges. Indicate any places where you use L'Hôpital's rule.

a. $\dfrac{n^2}{2^n}$

$$\lim_{n\to\infty} \frac{n^2}{2^n} \overset{\text{L'H}}{=} \lim_{n\to\infty} \frac{2n}{(\ln 2)\, 2^n} \overset{\text{L'H}}{=} \lim_{n\to\infty} \frac{2}{(\ln 2)^2 2^n}$$

$$\overset{\text{"}\frac{2}{\infty}\text{"}}{=} 0 \quad \underline{\text{converges}}$$

b. $a_n = \left(1 - \dfrac{2}{n}\right)^n$

$$\ln a_n = n \ln\left(1 - \frac{2}{n}\right) = \frac{\ln\left(1-\frac{2}{n}\right)}{\frac{1}{n}}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\ln a_n} = e^{-2}$$

$$\lim_{n\to\infty} \ln a_n \overset{\text{L'H}}{=} \lim_{n\to\infty} \frac{\frac{1}{1-2/n}\cdot +\frac{2}{\cancel{n^2}}}{-\frac{1}{\cancel{n^2}}} = \lim_{n\to\infty} \frac{2}{\frac{2}{n}-1} = \frac{2}{0-1} = -2$$

4. [3 points] Find a formula for the nth term a_n of this recursively-defined sequence, for $n \geq 1$:

$$a_1 = 3 \quad \text{and} \quad a_{n+1} = \frac{a_n}{n}.$$

$$a_2 = \frac{a_1}{1} = \frac{3}{1}$$

$$a_3 = \frac{a_2}{2} = \frac{3}{2\cdot 1}$$

$$a_4 = \frac{a_3}{3} = \frac{3}{3\cdot 2\cdot 1}$$

$$a_5 = \frac{a_4}{4} = \frac{3}{4\cdot 3\cdot 2\cdot 1}$$

it is helpful to <u>not</u> simplify until you see the pattern

$$a_n = \frac{3}{(n-1)!}$$

5. [6 points] Does the series converge or diverge? If it converges find its sum; if it diverges explain why.

a. $1+\frac{e}{\pi}+\frac{e^2}{\pi^2}+\frac{e^3}{\pi^3}+\dots$ $= 1 + 1\cdot\left(\frac{e}{\pi}\right) + 1\cdot\left(\frac{e}{\pi}\right)^2 + 1\cdot\left(\frac{e}{\pi}\right)^3 + \dots$

geometric series with $a=1$ and $r=\frac{e}{\pi}$

note $e<\pi$ so $\frac{e}{\pi}<1$ converges

$$1+\frac{e}{\pi}+\frac{e^2}{\pi^2}+\dots = \frac{a}{1-r} = \frac{1}{1-\frac{e}{\pi}} = \frac{\pi}{\pi-e}$$

b. $\sum_{n=1}^{\infty}\frac{n+1}{n}$ $= \frac{2}{1}+\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\dots$

$$\geq 1+1+1+1+1+\dots = \infty$$

$(S_k \geq k)$

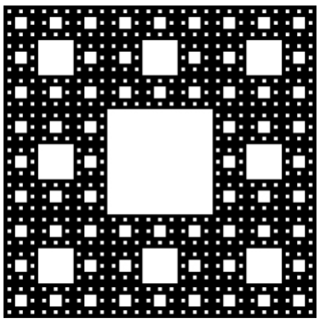
so diverges

6. [2 points] Compute and simplify the partial sum S_4 for the series in **5 b.**

$$S_4 = \frac{2}{1}+\frac{3}{2}+\frac{4}{3}+\frac{5}{4} = \frac{7}{2}+\frac{16+15}{12} = \frac{7}{2}+\frac{31}{12}$$

$$= \frac{42+31}{12} = \frac{73}{12}$$

Extra Credit. [1 point] The thing below is called the **Sierpinksi gasket**. It is built by removing the white parts from a black square. Assume the original square has side-length one and thus area one. Remove the middle 1/9th of the area. The remainder is 8 smaller black squares. For each of these, remove the middle 1/9th. Continuing in this way forever, you remove all the white area. Using geometric series, compute the white area you removed. What area is left, the black area?



$$\begin{pmatrix}\text{white}\\ \text{area}\end{pmatrix} = \frac{1}{9} + \frac{8}{9\cdot 9} + \frac{8\cdot 8}{9\cdot 9\cdot 9} + \dots$$

$$= \frac{1}{9} + \frac{1}{9}\left(\frac{8}{9}\right) + \frac{1}{9}\left(\frac{8}{9}\right)^2 + \dots$$

[geometric with $a = \frac{1}{9}$, $r = \frac{8}{9}$]

$$= \frac{a}{1-r} = \frac{1/9}{1-8/9} = \frac{1/9}{1/9} = 1$$

$$\begin{pmatrix}\text{black}\\ \text{area}\end{pmatrix} = 1 - \begin{pmatrix}\text{white}\\ \text{area}\end{pmatrix} = 1 - 1 = 0$$

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