SOLUTIONS

I. Compare the following three integrals. Which ones can a Calculus I student integrate? ← (a)? (b) ✓

(a) $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx$ (b) $\displaystyle\int \frac{x}{\sqrt{9-x^2}}\,dx$ (c) $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

(a) $$\int \frac{1}{\sqrt{9-x^2}}\,dx = \boxed{\arcsin\left(\frac{x}{3}\right)+C}$$ } from memory?!

by trig. sub. $= \displaystyle\int \frac{3\cos\theta\,d\theta}{\sqrt{9-9\sin^2\theta}} = \int \frac{3\cos\theta}{3\cos\theta}\,d\theta$

$\left[\begin{array}{l} x=3\sin\theta \\ dx=3\cos\theta\,d\theta \end{array}\right]$

$$= \int d\theta = \theta + C = \arcsin\left(\frac{x}{3}\right)+C$$

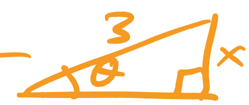
(b) $$\int \frac{x}{\sqrt{9-x^2}}\,dx = -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\frac{1}{2}\cdot 2u^{\frac{1}{2}}+C$$

$\left[\begin{array}{l} u=9-x^2 \\ du/-2 = x\,dx \end{array}\right]$ ↑ u-substitution!

$$= \boxed{-\sqrt{9-x^2}+C}$$

(c) $$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{9\sin^2\theta\cdot 3\cos\theta\,d\theta}{\sqrt{9-9\sin^2\theta}}$$

$\left[\begin{array}{l} x=3\sin\theta \\ dx=3\cos\theta\,d\theta \end{array}\right]$



$$=9\int \frac{\sin^2\theta\cos\theta}{\cos\theta}\,d\theta = 9\int \sin^2\theta\,d\theta = \frac{9}{2}\int 1-\cos(2\theta)\,d\theta$$

$$= \frac{9}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right)+C = \frac{9}{2}\left(\theta - \sin\theta\cos\theta\right)+C$$

(*)

$$= \boxed{\frac{9}{2}\left(\arcsin\left(\frac{x}{3}\right) - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right)+C}$$

Summary: If $\sqrt{a^2-x^2}$ appears in an integrand (**and** other techniques do not work), then

substitute $\boxed{x=a\sin\theta}$

(and use $1-\sin^2\theta=\cos^2\theta$)

II. Compare the following integrals. Which one can a Calculus I student integrate? ← (a)?

(a) $\int \frac{dx}{9+x^2}$ (b) $\int \frac{dx}{\sqrt{9+x^2}}$ (c) $\int \frac{dx}{\sqrt{x^2-9}}dx$

(a) $\int \frac{dx}{9+x^2} = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$ from memory?

$= \int \frac{3\sec^2\theta\, d\theta}{9+9\tan^2\theta} = \int \frac{3\sec^2\theta}{9\sec^2\theta}d\theta$

$\left[\begin{array}{l} x = 3\tan\theta \\ dx = 3\sec^2\theta\, d\theta \end{array}\right]$ ← by trig sub.

$= \frac{1}{3}\int d\theta = \frac{1}{3}\theta + C = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$

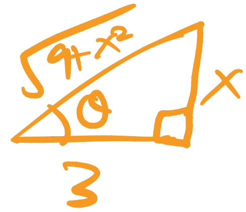
(b) $\int \frac{dx}{\sqrt{9+x^2}} = \int \frac{3\sec^2\theta\, d\theta}{\sqrt{9+9\tan^2\theta}} = \int \frac{3\sec^2\theta}{3\sec\theta}d\theta$

$\left[\begin{array}{l} x = 3\tan\theta \\ dx = 3\sec^2\theta\, d\theta \end{array}\right]$

from memory!

$= \int \sec\theta\, d\theta = \ln|\tan\theta + \sec\theta| + C$

$= \ln\left|\frac{x}{3} + \frac{\sqrt{9+x^2}}{x}\right| + C$



(c) on last page

Summary:

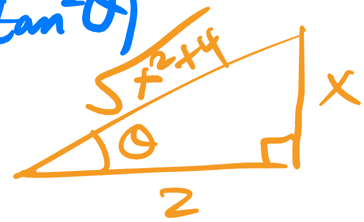
- If $\sqrt{a^2+x^2}$ appears in an integrand (**and** other techniques do not work), then substitute $\boxed{x = a\tan\theta}$
- If $\sqrt{x^2-a^2}$ appears in an integrand (**and** other techniques do not work), then substitute $\boxed{x = a\sec\theta}$

and use: $1+\tan^2\theta = \sec^2\theta$

III. Evaluate

(a) $\int \frac{dx}{(4+x^2)^2} = \int \frac{2\sec^2\theta\, d\theta}{(4+4\tan^2\theta)^2}$

$\begin{bmatrix} x = 2\tan\theta \\ dx = 2\sec^2\theta\, d\theta \end{bmatrix}$



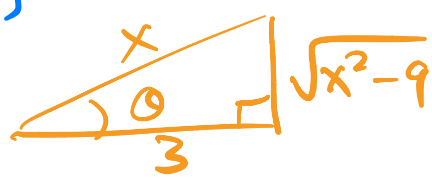
$= \int \frac{2\sec^2\theta\, d\theta}{16\sec^4\theta} = \frac{1}{8}\int \frac{d\theta}{\sec^2\theta} = \frac{1}{8}\int \cos^2\theta\, d\theta$

$= \frac{1}{16}\int 1+\cos(2\theta)\, d\theta = \frac{1}{16}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C$ (*)

$= \frac{1}{16}\left(\arctan\left(\frac{x}{2}\right) + \sin\theta\cos\theta\right) + C = \boxed{\frac{1}{16}\left(\arctan\left(\frac{x}{2}\right) + \frac{2x}{x^2+4}\right) + C}$

(b) $\int \frac{dx}{(x^2-9)^{3/2}} = \int \frac{3\sec\theta\tan\theta\, d\theta}{(9\sec^2\theta - 9)^{3/2}}$

$\begin{bmatrix} x = 3\sec\theta \\ dx = 3\sec\theta\tan\theta\, d\theta \end{bmatrix}$



$= \int \frac{3\sec\theta\tan\theta}{(3\tan\theta)^3}\, d\theta = \frac{1}{9}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$

$= \frac{1}{9}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\, d\theta = \frac{1}{9}\int \frac{\cos\theta\, d\theta}{\sin^2\theta}$

$= \frac{1}{9}\int \frac{du}{u^2} = -\frac{1}{9}u^{-1} + C = -\frac{1}{9}(\sin\theta)^{-1} + C$

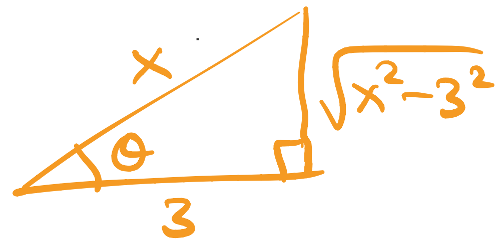
$\begin{bmatrix} u = \sin\theta \\ du = \cos\theta\, d\theta \end{bmatrix}$

$= -\frac{1}{9}\left(\frac{\sqrt{x^2-9}}{x}\right)^{-1} + C = \boxed{-\frac{x}{9\sqrt{x^2-9}} + C}$

II(c) $\displaystyle\int \frac{dx}{\sqrt{x^2-9}}$

$$\left[\begin{array}{l} x = 3\sec\theta \\ dx = 3\sec\theta\tan\theta\, d\theta \end{array}\right]$$

$$= \int \frac{3\sec\theta\tan\theta\, d\theta}{\sqrt{9\sec^2\theta - 9}}$$



$$= \int \frac{3\sec\theta\tan\theta}{3\tan\theta}\, d\theta = \int \sec\theta\, d\theta$$

$$= \ln|\tan\theta + \sec\theta| + C$$

$$= \ln\left|\frac{\sqrt{x^2-9}}{3} + \frac{x}{3}\right| + C$$

(*) a frequently-used trig. identity:

$$\boxed{\sin(2\theta) = 2\sin\theta\cos\theta}$$