

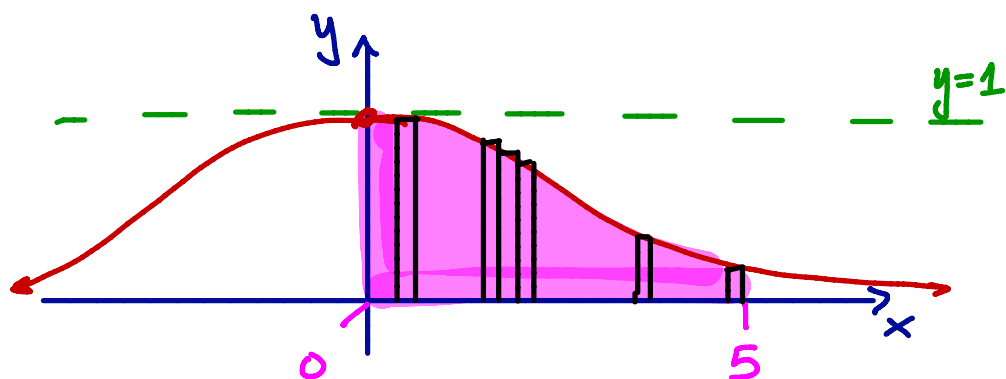
SECTION 2.1: AREA BETWEEN CURVES

1. Review from Calc 1

- (a) Set up an integral that evaluates the area under the curve $f(x) = \frac{1}{1+x^2}$ from $x = 0$ to $x = 5$.

$$\int_0^5 \frac{1}{1+x^2} dx$$

- (b) Sketch $f(x)$, shade the region from part (a). Include some sample Riemann rectangles that explain the formula from part (a).



• $f(0) = \frac{1}{1+0} = 1$

• as $x \rightarrow +\infty$,

$\frac{1}{1+x^2} \rightarrow 0$

• f is even.

area rectangle is (height) $\cdot$ (base)
 $f(x) \cdot dx$

So $\int_0^5 \frac{1}{1+x^2} dx$ is the sum ($\sum$) of areas of rectangles with height $f(x)$ and base dx from $x=0$ to $x=5$.

$\uparrow$ $\uparrow$
 $h \cdot b$
 $= \text{area}$

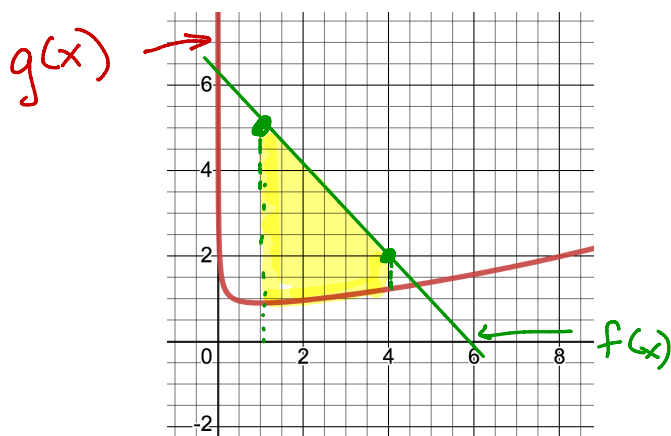
- (c) Can we evaluate the integral in part (a)? Explain. Is the answer plausible?

$$\int_0^5 \frac{1}{1+x^2} dx = \arctan(x) \Big|_0^5 = \arctan(5) - \arctan(0) = \arctan(5)$$

if you got 78.69,
 your calculator was set to degrees, not radians!

2. Introduction to Section 2.1 Areas Between Curves

- (a) The graph of the function $g(x) = \frac{e^{\sqrt{x}}}{3\sqrt{x}}$ is pictured below. On the same axes, sketch the graph of $f(x) = 6 - x$. Shade the area between $f(x)$ and $g(x)$ on the interval $[1, 4]$. Set up an integral (or integrals) that calculate(s) the area.



$$f(1) = 6 - 1 = 5 \quad (1, 5)$$

$$f(4) = 6 - 4 = 2 \quad (4, 2)$$

$$\text{area} = \int_1^4 \left[(6 - x) - \left(\frac{e^{\sqrt{x}}}{3\sqrt{x}} \right) \right] dx = *$$

yes! easy u-sub.

- (b) Can we evaluate the integral(s) above? (Do the integration if time permits.)

$$\int_1^4 (6 - x) dx = 6x - \frac{1}{2}x^2 \Big|_1^4 = (24 - 8) - (6 - \frac{1}{2}) = 10.5 = \frac{21}{2}$$

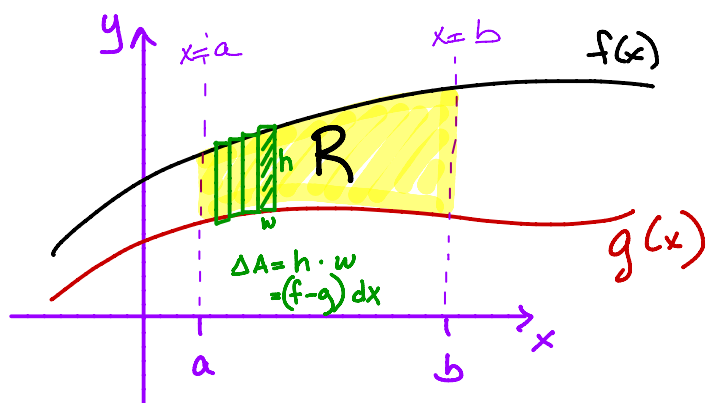
$$\frac{1}{3} \int_1^4 e^{x^{1/2}} x^{-1/2} dx = \frac{1}{3} \cdot 2 \int_1^2 e^u du = \frac{2}{3} e^u \Big|_1^2 = \frac{2}{3} (e^2 - e)$$

let $u = x^{1/2}$
 $du = \frac{1}{2} x^{-1/2} dx$
 $2 du = x^{-1/2} dx$

If $x = 1, u = 1$
 $x = 4, u = 2$

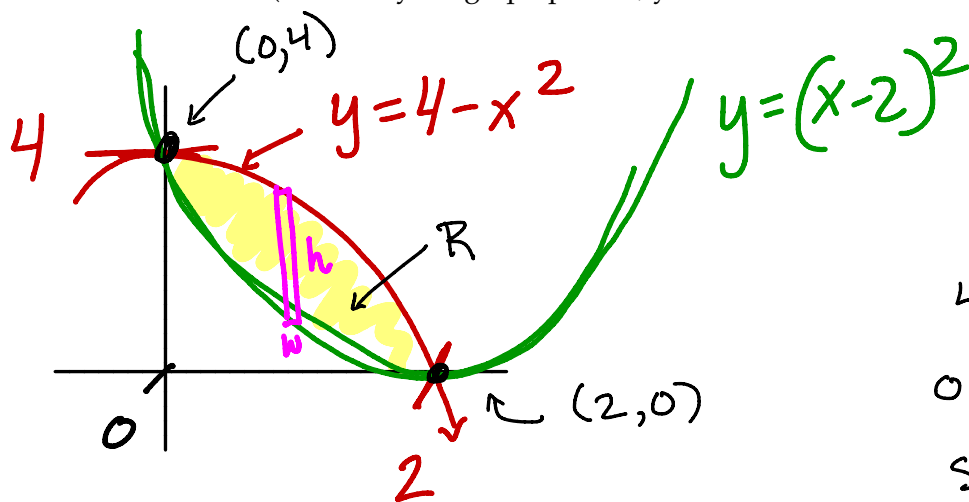
$$* = \frac{21}{2} - \frac{2}{3} (e^2 - e) \approx 7.386$$

3. Theorem 2.1: Let $f(x)$ and $g(x)$ be continuous on $[a, b]$ and $f(x) \geq g(x)$ always. Then the area of the region R bounded above by $f(x)$, below by $g(x)$, on the left by $x = a$, and on the right by $x = b$, is given by



$$\text{area}(R) = A = \int_a^b (f(x) - g(x)) dx$$

4. (a) Sketch the graphs of $h(x) = 4 - x^2$ and $k(x) = (x - 2)^2$ and shade the region R bounded by them. (To make your graph precise, you will need to determine the points of intersection.)



pts of intersection

$$4 - x^2 = (x - 2)^2$$

$$4 - x^2 = x^2 - 4x + 4$$

$$0 = 2x^2 - 4x = 2x(x - 2)$$

So $x = 0$ or $x = 2$

- (b) Set up an integral to find the area of region R . (You will need to determine the points of intersection.)

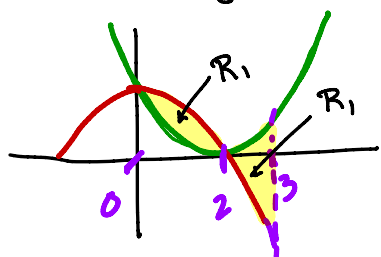
$$A = \int_0^2 \underbrace{[(4 - x^2) - ((x - 2)^2)]}_{h} \underbrace{dx}_{w} \quad \Rightarrow \quad \int_0^2 (4 - x^2 - (x^2 - 4x + 4)) dx$$

- (c) Can we evaluate the integral above? (Do the integration if time permits.)

$$= \int_0^2 (-2x^2 + 4x) dx = \left[-\frac{2}{3}x^3 + 2x^2 \right]_0^2 = -\frac{2}{3} \cdot 8 + 2 \cdot 4 = -\frac{16}{3} + \frac{24}{3} = \frac{8}{3}$$

- (d) How would the ^{integral}~~problem~~ change if you were asked to find the area between $h(x)$ and $k(x)$ on the interval $[0, 3]$?

The graphs cross at $x = 2$. The region is split into two pieces with different top/bottom bounds.



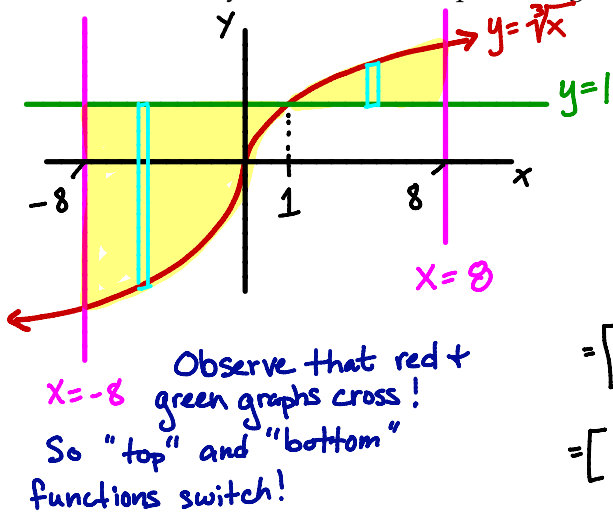
$$\text{area} = \text{area}(R_1) + \text{area}(R_2)$$

$$= \int_0^2 (4 - x^2 - (x - 2)^2) dx + \int_2^3 ((x - 2)^2 - (4 - x^2)) dx$$

just like 4b $\uparrow$ $\frac{4b}{3}$ $\uparrow$ new piece switches bounds

Variations on a Theme

4. Sketch the region bounded by $y = x^{1/3}$, $y = 1$, $x = -8$ and $x = 8$. Include a sample rectangle in your sketch. Set up an integral to calculate its area. Evaluate the integral, if time permits.



$$A = \int_{-8}^1 \left(1 - x^{\frac{1}{3}}\right) dx + \int_1^8 \left(x^{\frac{1}{3}} - 1\right) dx$$

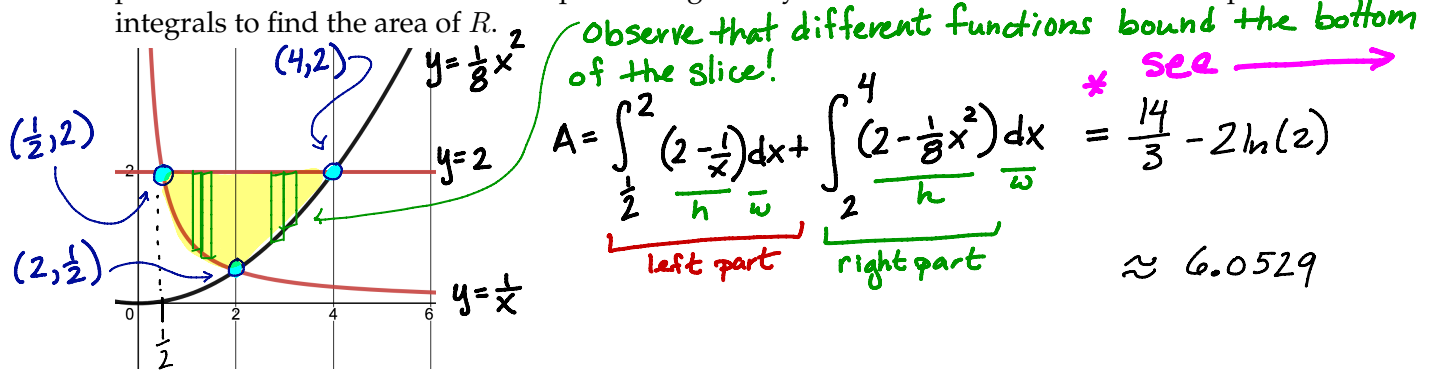
$$= \left[x - \frac{3}{4} x^{\frac{4}{3}} \right]_{-8}^1 + \left[\frac{3}{4} x^{\frac{4}{3}} - x \right]_1^8$$

$$= \left[\left(1 - \frac{3}{4}\right) - \left(-8 - \frac{3}{4}(16)\right) \right] + \left[\left(\frac{3}{4} \cdot 16 - 8\right) - \left(\frac{3}{4} - 1\right) \right]$$

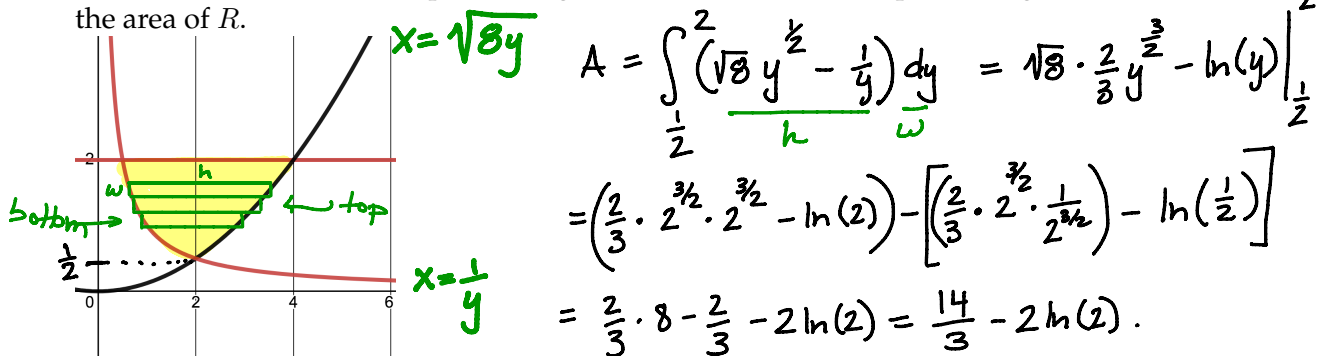
$$= \left[\frac{1}{4} + 20 \right] + \left[4 + \frac{1}{4} \right] = 24.5$$

5. Both problems below concern the same region, R (sketched below), bounded on three sides by $y = 2$, $y = \frac{1}{x}$, and $y = \frac{x^2}{8}$.

- (a) On the graph below, label each curve with its algebraic formula and the coordinates of each point of intersection. Include sample rectangles in your sketch and use them to set up two integrals to find the area of R .



- (b) On the graph below, label each curve with its algebraic formula solved for x instead of y , if possible. Sketch horizontal sample rectangles and use them to set up one integral that calculates the area of R .



- (c) What are the pros and cons of using vertical versus horizontal slices?

$$* \int_{\frac{1}{2}}^2 \left(2 - \frac{1}{x}\right) dx + \int_2^4 \left(2 - \frac{1}{8}x^2\right) dx = \left(2x - \ln(x)\right) \Big|_{\frac{1}{2}}^2 + \left(2x - \frac{1}{24}x^3\right) \Big|_2^4$$

$$= \left(4 - \ln(2)\right) - \left(1 - \ln\left(\frac{1}{2}\right)\right) + \left(8 - \frac{64}{24}\right) - \left(4 - \frac{8}{24}\right)$$

$$= 3 - 2\ln(2) + 4 - \frac{56}{24}$$

$$= 4\frac{2}{3} - \ln(2) = \frac{14}{3} - 2\ln(2) \approx 6.0529$$