

SECTION 3.7: IMPROPER INTEGRALS (DAY 2)

Compute these integrals with friends! Please carefully write the limit, for example

$$\int_1^{\infty} \frac{dx}{x^2} = \lim_{t \rightarrow \infty} \int_1^t \frac{dx}{x^2} = \lim_{t \rightarrow \infty} \left[-\frac{1}{x} \right]_1^t = \lim_{t \rightarrow \infty} 1 - \frac{1}{t} = 1$$

$$1. \int_2^{\infty} \frac{1}{9+x^2} dx = \lim_{b \rightarrow \infty} \left(\frac{1}{9} \int_2^b \frac{dx}{1+(\frac{x}{3})^2} \right) = \lim_{b \rightarrow \infty} \left(\frac{1}{3} \arctan\left(\frac{x}{3}\right) \Big|_2^b \right)$$

as $b \rightarrow \infty$
 $\arctan\left(\frac{b}{3}\right) \rightarrow \frac{\pi}{2}$

$$= \lim_{b \rightarrow \infty} \left(\frac{1}{3} \left(\arctan \frac{b}{3} - \arctan \frac{2}{3} \right) \right) = \frac{1}{3} \left(\frac{\pi}{2} - \arctan\left(\frac{2}{3}\right) \right)$$

Converges

$$2. \int_{-\infty}^0 e^x dx = \lim_{a \rightarrow -\infty} \left(\int_a^0 e^x dx \right) = \lim_{a \rightarrow -\infty} \left(e^x \Big|_a^0 \right) = \lim_{a \rightarrow -\infty} (e^0 - e^a) = \lim_{a \rightarrow -\infty} (1 - e^a)$$

as $a \rightarrow -\infty$
 $e^a \rightarrow 0$



$$= 1 - 0 = 1$$

Converges

$$3. \int_0^1 \frac{1}{\sqrt[4]{x}} dx = \lim_{a \rightarrow 0^+} \left(\int_a^1 x^{-1/4} dx \right) = \lim_{a \rightarrow 0^+} \left(\frac{4}{3} x^{3/4} \Big|_a^1 \right) = \lim_{a \rightarrow 0^+} \left(\frac{4}{3} (1 - a^{3/4}) \right)$$

$$= \frac{4}{3} (1 - 0) = \frac{4}{3}$$

Converges

$$4. \int_0^1 \ln t dt = \lim_{a \rightarrow 0^+} \left(\int_a^1 \ln t dt \right) = \lim_{a \rightarrow 0^+} \left(t \ln t - t \Big|_a^1 \right)$$

aside: (IBP)

$$\int \ln t dt = t \ln t - \int dt = t \ln t - t$$

$u = \ln t \quad dv = dt$
 $du = \frac{1}{t} dt \quad v = t$

$$= \lim_{a \rightarrow 0^+} \left((1 \cdot \ln(1) - 1) - (a \ln(a) - a) \right)$$

$$= \lim_{a \rightarrow 0^+} \left(-1 - a \ln(a) + a \right)$$

$$= -1 - 0 + 0 = -1$$

See attached sheet!

Converges

L'Hôpital's Rule to the rescue

$$\lim_{a \rightarrow 0^+} \boxed{a \ln a} = \lim_{a \rightarrow 0^+} \frac{\ln a}{a^{-1}} \stackrel{④}{=} \lim_{a \rightarrow 0^+} \frac{\frac{1}{a}}{-a^{-2}} = \lim_{a \rightarrow 0^+} \frac{1}{a} \cdot \frac{-a^2}{1}$$

$$= \lim_{a \rightarrow 0^+} -a = 0.$$

$$5. \int_1^2 \frac{dx}{1-x} = \lim_{a \rightarrow 1^+} \left(\int_a^2 \frac{dx}{1-x} \right) = \lim_{a \rightarrow 1^+} \left(-\ln|1-x| \Big|_a^2 \right) = \lim_{a \rightarrow 1^+} \left(-\ln|1| + \ln|1-a| \right)$$

$$= \lim_{a \rightarrow 1^+} \left(\ln|1-a| \right) = \lim_{a \rightarrow 1^+} \ln(a-1) = -\infty \quad \text{diverges}$$

↓
If exponent is zero or negative, then it converges. Otherwise, not.

$$6. \int_0^\infty e^x e^{-sx} dx = \lim_{b \rightarrow \infty} \left(\int_0^b e^x e^{-sx} dx \right) = \lim_{b \rightarrow \infty} \left(\int_0^b e^{(1-s)x} dx \right)$$

$$= \lim_{b \rightarrow \infty} \left(\frac{1}{1-s} \cdot e^{(1-s)x} \Big|_0^b \right) = \lim_{b \rightarrow \infty} \left(\frac{e^{(1-s)b}}{1-s} - \frac{1}{1-s} \right) = \begin{cases} \frac{1}{s-1} & \text{if } 1 \leq s \\ \infty & \text{if } 1 > s \end{cases}$$

Converges for some s values and diverges for others

$$7. \int_{-1}^8 \frac{1}{x^{4/3}} dx = \lim_{t \rightarrow 0^-} \int_{-1}^t x^{-4/3} dx + \lim_{t \rightarrow 0^+} \int_t^8 x^{-4/3} dx \quad \left[\text{If one of these diverges, the whole diverges. I look @ first.} \right]$$

$$\bullet \lim_{t \rightarrow 0^-} \int_{-1}^t x^{-4/3} dx = \lim_{t \rightarrow 0^-} \left(-3x^{-1/3} \Big|_{-1}^t \right) = \lim_{t \rightarrow 0^-} -3 \left(\frac{1}{t^{1/3}} - \frac{1}{(-1)^{1/3}} \right) = \infty$$

because as $t \rightarrow 0^-$, $\frac{1}{t^{1/3}} \rightarrow -\infty$. DIVERGES

$$8. \int_2^\infty \frac{dx}{x \ln^3 x} = \lim_{b \rightarrow \infty} \left(\int_2^b \frac{(\ln x) dx}{x} \right) = \lim_{b \rightarrow \infty} \left(-\frac{1}{2} (\ln x)^{-2} \Big|_2^b \right)$$

$$= \lim_{b \rightarrow \infty} \left(-\frac{1}{2} \left(\frac{1}{(\ln b)^2} - \frac{1}{(\ln 2)^2} \right) \right) = -\frac{1}{2} \left(0 - \frac{1}{(\ln 2)^2} \right) = \frac{1}{2(\ln 2)^2}$$

Converges