1. Example 4b from §5.2 Day 1 is an example of a *telescoping series* which means (most) of the middle terms cancel leave a handful at beginning and end.

(#4b) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \underbrace{\left(\frac{1}{1}-\frac{1}{2}\right)}_{n=1}+\underbrace{\left(\frac{1}{2}-\frac{1}{3}\right)}_{n=2}+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots+\left(\frac{1}{n}-\frac{1}{n+1}\right)+\cdots$

$\boxed{\text{partial fraction} \quad \frac{1}{n(n+1)} = \frac{1}{n}-\frac{1}{n+1}}$

So $S_n = 1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\dots+\frac{1}{n}-\frac{1}{n+1} = 1-\frac{1}{n+1}$; $\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1-\frac{1}{n+1} = 1$

So the series converges to 1

2. **(⋆)** A **geometric series** has form $\sum_{n=1}^{\infty} ar^{n-1} = ar^0 + ar + ar^2 + \dots + ar^{n-1} + \dots$
$= a + ar + ar^2 + \dots + ar^{n-1} + \dots$

If $|r|<1$, then $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$. (It converges.)

If $|r|\geq 1$, then $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.

Why? → * end

If $r=1$ or $r=-1$, this is easy to see. For the rest

3. For each series below, determine whether or not they converge. (Justify your conclusion.) If they do converge, determine their sum.

(a) $\sum_{n=1}^{\infty} \pi\left(\frac{2}{3}\right)^{n-1} = \frac{\pi}{1-\frac{2}{3}} = 3\pi$. converges

$a=\pi$

$r=\frac{2}{3}$, $|r|<1$

(b) $\sum_{n=1}^{\infty} \frac{4^{n-1}}{3^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{3^2}\left(\frac{4}{3}\right)^{n-1}$. geometric series with $|r| = \left|\frac{4}{3}\right| \geq 1$

$a=\frac{1}{9}$, $r=\frac{4}{3}>1$ So the series diverges.

(c) $\sum_{n=1}^{\infty} 10\left(\frac{-3}{5}\right)^n = \sum_{n=1}^{\infty} 10\left(\frac{-3}{5}\right)\left(\frac{-3}{5}\right)^{n-1} = \frac{-6}{1-(\frac{-3}{5})} = \frac{-6}{\frac{8}{5}} = -6\cdot\frac{5}{8} = \frac{-15}{4}$

$a = -6$

$r = \frac{-3}{5}$

4. **($\star$) The harmonic series** is $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} + \dots$. $\boxed{\text{It diverges.}}$

- $\sum_{n=1}^{\infty} \frac{1}{2} = \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2} + \dots$ diverges b/c $S_n = \frac{1}{2}n$ and $\lim_{n\to\infty} \frac{1}{2}n = \infty$.

- Now $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \dots + \frac{1}{8}} + \underbrace{\frac{1}{9} + \frac{1}{10} + \dots + \frac{1}{16}} + \underbrace{\frac{1}{17} + \dots + \frac{1}{32}} + \dots$

$> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \dots + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{16} + \frac{1}{32} + \dots + \frac{1}{32} + \frac{1}{64} + \dots + \frac{1}{64}$

$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$

Since $\sum \frac{1}{2}$ diverges and $\sum \frac{1}{n}$ is LARGER, the series $\sum \frac{1}{n}$ diverges.

5. Extra Practice

(a) $\sum_{n=1}^{\infty}(e^{2/n} - e^{2/(n+1)}) = (e^2 - e^1) + (e^1 - e^{2/3}) + (e^{2/3} - e^{2/4}) + (e^{2/4} - e^{2/5}) + \dots$

$S_n = e^2 - e^{2/n+1}$; $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(e^2 - e^{2/n+1}\right) = e^2 - e^0 = e^2 - 1$

(b) $\sum_{n=1}^{\infty}\left[\left(\frac{-1}{2}\right)^{n-1} + \left(\frac{4}{5}\right)^n\right] = \frac{2}{3} + 4 = \frac{14}{3}$

$\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^{n-1} = \frac{1}{1-(-\frac{1}{2})} = \boxed{\frac{2}{3}}$

$\sum_{n=1}^{\infty} \frac{4}{5}\left(\frac{4}{5}\right)^{n-1} = \frac{4/5}{1-\frac{4}{5}} = \frac{4}{5}\cdot\frac{5}{1} = \boxed{4}$

* why? (Suppose $|r| \neq 1$)

For $\sum_{n=1}^{\infty} ar^{n-1}$, $S_n = a + ar + ar^2 + \dots + ar^{n-1}$.

So... $rS_n = r(a + ar + ar^2 + \dots + ar^{n-1})$

$= ar + ar^2 + ar^3 + \dots + ar^n$

Now

$$\begin{array}{rl} & S_n = a + ar + ar^2 + \dots \qquad + ar^{n-1} \\ -(& rS_n = \quad ar + ar^2 + ar^3 + \dots + ar^{n-1} + ar^n) \\ \hline & S_n - rS_n = a \qquad\qquad - ar^n \end{array}$$

Sub-tract!

Or, $S_n = \dfrac{a - ar^n}{1 - r}$ → why $|r| \neq 1$

Apply the definition: $\sum_{n=1}^{\infty} ar^{n-1}$ converges if $\lim_{n\to\infty} S_n = L$.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a - ar^n}{1 - r} = \lim_{n\to\infty} \left(\frac{a}{1-r}\right)(1 - r^n)$$

$$= \left(\frac{a}{1-r}\right) \lim_{n\to\infty} (1 - r^n) = \begin{cases} \dfrac{a}{1-r} & \text{for } |r| < 1 \\ \text{diverges} & \text{for } |r| \geq 1 \end{cases}$$

If $|r| < 1$, $\lim_{n\to\infty} r^n = 0$.

If $|r| > 1$, $\lim_{n\to\infty} r^n$ diverges.