(1) An *alternating series* has the form

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \dots$$

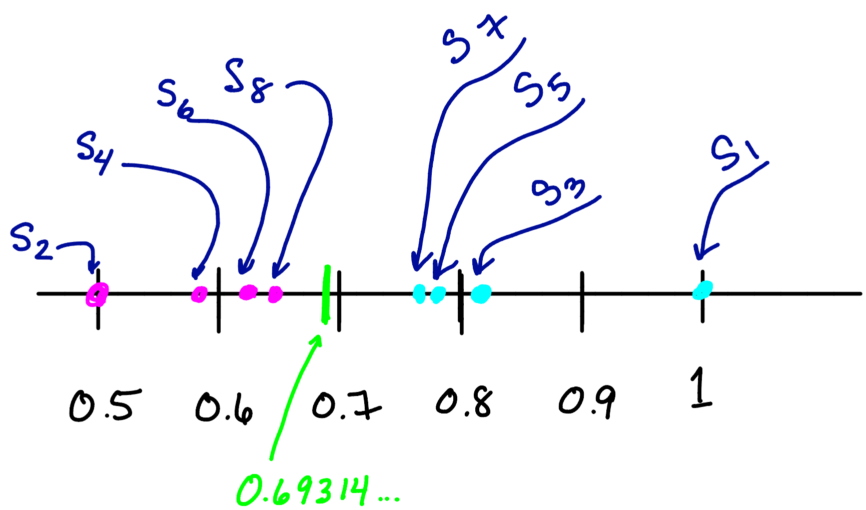
$$\sum_{n=1}^{\infty} (-1)^{n} b_n = -b_1 + b_2 - b_3 + b_4 + \dots$$

(2) Example A: (!!) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \dots$. alternating harmonic series

What is b_n? Ans: $\frac{1}{n}$

Does it converge? Diverge?

- $S_1 = 1$
- $S_2 = 1 - \frac{1}{2} = 0.5$
- $S_3 = 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6} = 0.8\overline{3}$
- $S_4 = \frac{5}{6} - \frac{1}{4} = 0.58\overline{3}$
- $S_5 = 0.783$
- $S_6 = 0.61\overline{6}$
- $S_7 = 0.7595$
- $S_8 = 0.6345$



- monotone, bounded
- monotone, bounded

(3) The Alternating Series Test

Given $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ or $\sum_{n=1}^{\infty} (-1)^{n} b_n$, the series will converge

if ① $\lim_{n\to\infty} b_n = 0$ and ② $b_n \geq b_{n+1} > 0$

Apply to alt. harmonic.

① $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

② Is $b_n = \frac{1}{n} \geq \frac{1}{n+1} = b_{n+1}$?

ANS: Know $n+1 > n$.

So $b_{n+1} = \frac{1}{n+1} < \frac{1}{n} = b_n$ ✓

Are they really alternating? Yes

(4) Determine whether the alternating series below converge or diverge. Justify your conclusion.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{3n+1}$

$b_n = \frac{n}{3n+1}$

Not in b_n !

① $\lim_{n\to\infty} \frac{n}{3n+1} = \frac{1}{3}$. Failed first step.

This always leads to this.

Apply the Divergence Test!

$\lim_{n\to\infty} \frac{(-1)^{n-1} n}{3n+1} = \text{DNE}$. So the series diverges.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{2^n}$

$b_n = \frac{n}{2^n}$

✓ ① $\lim_{n\to\infty} \frac{n}{2^n} \overset{LH}{=} \lim_{n\to\infty} \frac{1}{\ln 2 \cdot 2^n} = 0$

✓ ② Is $b_n = \frac{n}{2^n} \geq \frac{n+1}{2^{n+1}} = b_{n+1}$?

Work: want $\boxed{\frac{n}{2^n} \geq \frac{n+1}{2^{n+1}}} \equiv \boxed{\frac{2^{n+1}}{2^n} \geq \frac{n+1}{n}} \equiv \boxed{2 \geq 1 + \frac{1}{n}}$

This is true for all $n \geq 1$.

So $b_n \geq b_{n+1}$.

So, by the A.S.T, $\sum \frac{(-1)^{n-1} n}{2^n}$ converges. ☺

(5) Remainders in Alternating Series and How to Estimate Them

Given $S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$, a convergent alt. series

$S_N = N^{th}$ partial sum
$= \sum_{n=1}^{N} (-1)^{n-1} b_n$
$=$ estimate of S

The point

$|R_N| \leq b_{N+1}$

$S_N \approx S$.

How good?

R_N = remainder = error
$= S - S_N$

(6) Show the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges. Then determine how large k needs to be so that the kth partial sum, S_k, is within $0.0001 = 10^{-4}$ of the sum of the series?

Convergence: Apply A.S.T. ① $\lim_{n\to\infty} \frac{1}{n^2} = 0$ and ② $\frac{1}{n^2} > \frac{1}{(n+1)^2}$

So, by AST, $\sum \frac{(-1)^{n+1}}{n^2}$ converges

S_3, R_3, interpretation:

- $S_3 = 1 - \frac{1}{4} + \frac{1}{9} = 0.86\overline{1}$
- $|R_3| \leq \frac{1}{16} \approx 0.0625$

- Interpretation We are estimating the sum of the series as $S_3 = 0.86\overline{1}$. This estimate is within 0.0625 of the actual value.

How large does k need to be?

We need k so that $|R_k| \leq 10^{-4}$. Since $|R_k| \leq b_{k+1}$, we need $b_{k+1} = \frac{1}{(k+1)^2} < \frac{1}{10^4}$ or $10^4 < (k+1)^2$. So $100 < k+1$.

So we need k to be at least 102. That is, to estimate the sum to within $\frac{1}{10000}$ of the correct value, we need 102 terms.

(7) **Definitions:** Absolute and Conditional Convergence

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

If $\sum_{n=1}^{\infty} |a_n|$ diverges and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n$ is conditionally convergent.

(8) For each series below, determine if the series is absolutely convergent, conditionally convergent, or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$

Check for absolute convergence

$\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$, a convergent p-series.

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$

Check for absolute convergence

$\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{2n+1} \right| = \sum_{n=1}^{\infty} \frac{1}{2n+1}$. Apply L.C.T. using $\sum_{n=1}^{\infty} \frac{1}{n}$, a divergent series.

$\displaystyle\lim_{n\to\infty} \frac{1}{2n+1} \cdot \frac{n}{1} = \frac{1}{2} \neq 0$. So $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+1}$ diverges.

So the series is not absolutely convergent.

Check for convergence

Apply the A.S.T. $b_n = \frac{1}{2n+1}$.

① $\displaystyle\lim_{n\to\infty} \frac{1}{2n+1} = 0$ ✓ ② $b_n = \frac{1}{2n+1} > \frac{1}{2n+3} = b_{n+1}$ ✓

So the series $\sum \frac{(-1)^{n-1}}{2n+1}$ converges.

The series $\sum \frac{(-1)^{n-1}}{2n+1}$ is conditionally convergent