(1) Find the Maclaurin Series for $f(x) = e^x$. So, the Taylor Series with $a = 0$.

$f(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x$

$\vdots$

$f^{(n)}(x) = e^x$

at $x = 0$

$f^{(n)}(x) = e^0 = 1$

So $f(x) = e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

I.O.C : $(-\infty, \infty)$

(2) Find $p_5(x)$, the 5th Taylor polynomial for $f(x) = e^x$.

$$e^x \approx p_5(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$$

(3) Use $p_5(x)$ to estimate the value of $e^2$ and determine the error.

$$p_5(2) = 1 + 2 + \frac{2^2}{2} + \frac{1}{6}2^3 + \frac{1}{24}2^4 + \frac{1}{120}2^5 \approx 7.26666..$$

$e^2 = 7.389056..$

error $= |e^2 - p_5(2)| = 0.122389...$ $\leq 0.8$? Yes!

What would this indicate in this case..??

Use $M \approx 9$, $a = 0$, $n = 5$, $x = 2$

So $|R_5| \leq \frac{9 \cdot 2^6}{6!} = 0.8$

(4) **Taylor's Remainder Theorem**

- $f(x)$ – function w/ lots of derivatives
- $p_n(x)$ – $n^{th}$ Taylor poly centered at $x=a$
- $I$ – interval containing $x=a$
- $R_n(x) = f(x) - p_n(x)$ = error or remainder
- $M$ the maximum value of $f^{(n+1)}(x)$ on $I$

Then $|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$

How large would we need $n$ to be in order to know $|R_n| < \frac{1}{100}$?

We need $\frac{9}{(n+1)!} \cdot 2^{n+1} \leq 0.01$

Start checking $n$-values

| $n$ | $9 \cdot 2^{n+1}/(n+1)!$ |
|---|---|
| 6 | 0.228 |
| 8 | 0.01269 |
| 9 | 0.00253 ✓ |

So if we want to approximate $e^2$ to within $\frac{1}{100}$ th of the correct value, we need $p_9(x)$, the $9^{th}$ Taylor polynomial.

(5) (HW 6.3 # 131)

(a) Why have you never been asked to evaluate $\int_0^1 e^{-x^2}\,dx$?

We don't know a function $F(x)$ such that $F'(x) = e^{-x^2}$.

Specifically if $h(x) = e^{-x^2}$, $h'(x) = -2xe^{-x^2}$.

If $h(x) = \frac{e^{-x^2}}{-2x}$, then $h'(x) = \frac{e^{-x^2}}{2x^2} + e^{-x^2}$

(b) **Use the Maclaurin Series for** $f(x) = e^x$ to find the Maclaurin Series for $g(x) = e^{-x^2}$.

From page 1, we know $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots$

We find the series for $g(x) = e^{-x^2}$ by plugging $-x^2$ in for $x$:

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$$

(c) Use $p_6(x)$, the 6th Maclaurin Series for $g(x) = e^{-x^2}$ to estimate $\int_0^1 e^{-x^2}\,dx$.

$$\text{So } \int_0^1 e^{-x^2}\,dx \approx \int_0^1 \left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{x^8}{24}\right)dx = x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \frac{1}{216}x^9 \Big|_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} = 0.7474867\ldots$$

(d) How good is your estimation?

Plug int computational tool: 0.746824