Section 7.2: Calculus of Parametric Curves

(1) Suppose you are given a curve defined as $x(t)$ and $y(t)$:

(a) $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

(b) $\frac{d^2y}{dx^2} = \frac{d}{dt}\left[{}^{dy}/_{dx} \right] \Big/ \frac{dx}{dt}$

(2) Given the parametric equations $x(t) = t^3 + 1,\ y(t) = 2t - t^2$

(a) Find dy/dx and determine its value at $t = -1, 0$, and 1.

$$\frac{dy}{dx} = \frac{(2-2t)}{3t^2} = \frac{2}{3}(t^{-2} - t^{-1})$$

$$\left.\frac{dy}{dx}\right|_{t=-1} = \frac{2}{3}\left(\frac{1}{(-1)^2} - \frac{1}{-1}\right) = \frac{2}{3}(1+1) = \frac{4}{3}$$

$$\left.\frac{dy}{dx}\right|_{t=0} = \text{DNE}, \quad \left.\frac{dy}{dx}\right|_{t=1} = \frac{2}{3}\left(\frac{1}{1} - \frac{1}{1}\right) = 0$$

$m = 0$

(b) Write the equation of the tangent line to the curve at $t = 1$.

$y(1) = 2 \cdot 1 - 1 = 1$

$x(1) = 1^3 + 1 = 2$

$y - 2 = 0(x-2)$ or $\boxed{y=2}$

(c) Find d^2y/dx^2.

$$\frac{d}{dt}\left[\frac{2}{3}(t^{-2} - t^{-1})\right]$$

$$= \frac{2}{3}(-2t^{-3} + t^{-2})$$

$$\text{So } \frac{d^2y}{dx^2} = \frac{\frac{2}{3}(-2t^{-3} + t^{-2})}{3t^2} = \frac{2}{9t^2}\left(\frac{-2}{t^3} + \frac{1}{t^2}\right)$$

$$= \frac{2}{9t^4}\left(\frac{-2}{t} + 1\right)$$

(d) Is the curve concave up or concave down at $t = 1$?

$$\left.\frac{d^2y}{dx^2}\right|_{t=1} = \frac{2}{9}(-2+1) = -\frac{2}{9} < 0.$$ So it's concave down

(e) Use technology to graph the curve and see if your answers above are plausible.

$a = x(\alpha),\ b = x(\beta) \qquad x'(t) = \frac{dx}{dt}$

on $[a,b]$ ← an x-interval

(3) Suppose you are given a curve defined as $x(t)$ and $y(t)$ the area under curve can be calculated as:

$$A = \int_a^b f(x)\,dx = \int_a^b y\,dx = \int_{t=\alpha}^{t=\beta} y(t)\,x'(t)\,dt$$

(4) Given the parametric equations $x(t) = t^3 + 1,\ y(t) = 2t - t^2$, determine the interval of t-values for which the curve above the x-axis. For this interval, find the area below the curve and above the x-axis.

$0 = 2t - t^2 = t(2-t)$

So $t=0$ and $t=2$

$$\int_0^2 (2t-t^2)(3t^2)\,dt = \int_0^2 (6t^3 - 3t^4)\,dt = \frac{6}{4}t^4 - \frac{3}{5}t^5 \Big|_0^2$$

$$= \frac{3}{2}\cdot 2^4 - \frac{3}{5}2^5 = 16\left(\frac{3}{2} - \frac{6}{5}\right)$$

$$= 16\left(\frac{15-12}{10}\right) = \frac{48}{10} = \frac{24}{5}$$

from $[a,b]$ ← x-values

(5) Suppose you are given a curve defined as $x(t)$ and $y(t)$ the arc length of curve can be calculated as

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx = \int_\alpha^\beta \sqrt{(y'(t))^2 + (x'(t))^2}\,dt \qquad \begin{array}{l} a = x(\alpha) \\ b = x(\beta) \end{array}$$

(6) Determine the arc length of the cycloid $x(\theta) = \theta - \sin(\theta)$ and $y(\theta) = 1 - \cos(\theta)$ from $t = 0$ to $t = 2\pi$.

$x'(\theta) = 1 - \cos\theta$ * use $\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$ $\boxed{\text{OR}}$ $\underline{2\sin^2\theta = 1 - \cos(2\theta)}$

$y'(\theta) = \sin\theta$

$(x')^2 = (1-\cos\theta)^2$

$= 1 - 2\cos\theta + \cos^2\theta$

$(y')^2 = \sin^2\theta$

$(x')^2 + (y')^2 = 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta$

$= 2 - 2\cos\theta = 2\cdot(1-\cos(\theta))$

$* = 4\sin^2\left(\frac{\theta}{2}\right)$

$$L = \int_0^{2\pi} \sqrt{4\sin^2\left(\tfrac{\theta}{2}\right)}\,d\theta = \int_0^{2\pi} 2\sin\left(\tfrac{\theta}{2}\right)d\theta$$

$$= -2\cdot 2\cdot\cos\left(\tfrac{\theta}{2}\right)\Big|_0^{2\pi}$$

$$= -4\left(\cos(\pi) - \cos(0)\right)$$

$$= -4(-1-1) = 8 \checkmark$$