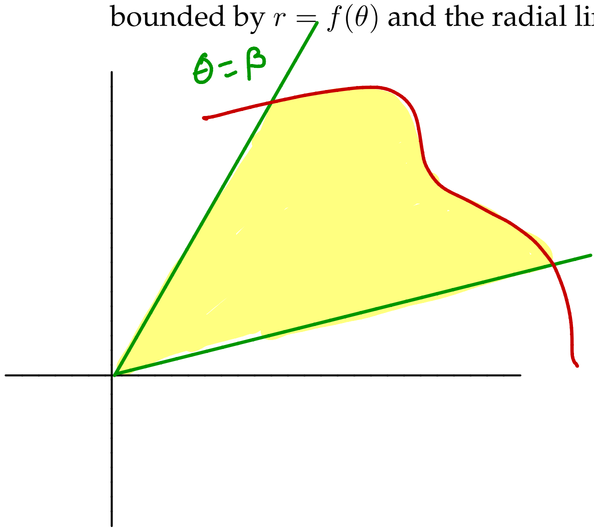
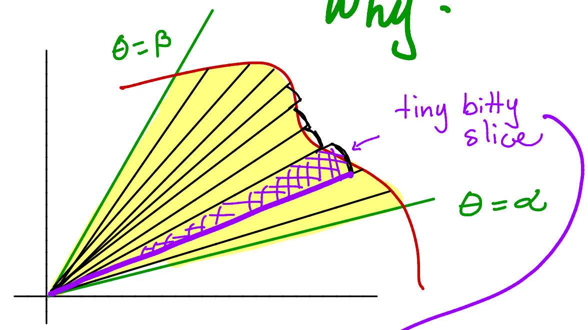
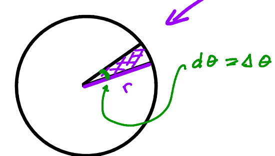
(1) Suppose $r = f(\theta)$ is a continuous and nonnegative on the interval from $\alpha \le \theta \le \beta$, then the area bounded by $r = f(\theta)$ and the radial lines $\theta = \alpha$ and $\theta = \beta$ is

Why?



$r = f(\theta)$



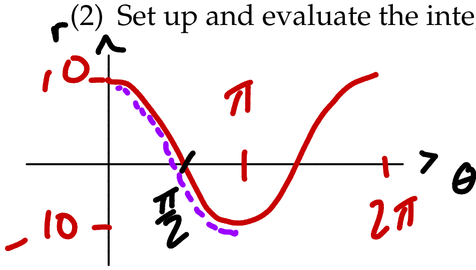


$$\frac{\boxed{\text{area sector}}}{\text{area circle}} = \frac{\Delta\theta}{2\pi} \quad \text{or} \quad \boxed{\text{area sector}} = \frac{(\text{area circle})}{2\pi}\, d\theta$$

$$\boxed{\text{area sector}} = \frac{\pi r^2}{2\pi}\, d\theta = \frac{1}{2} r^2\, d\theta$$

$$A = \int_\alpha^\beta \frac{1}{2}\left[f(\theta)\right]^2 d\theta$$

(2) Set up and evaluate the integral to find the area enclosed by the polar curve $r = 10\cos(\theta)$.

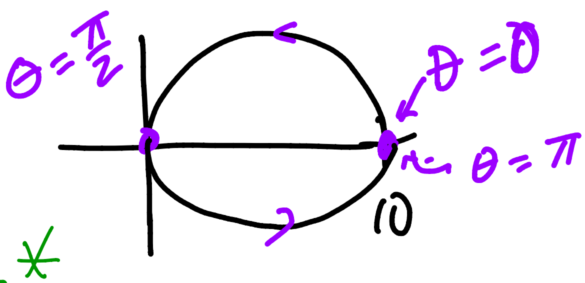


$$A = \int_0^\pi \frac{1}{2}\left(10\cos\theta\right)^2 d\theta = 50\int_0^\pi \cos^2\theta\, d\theta$$

$$= 25\int_0^\pi \left(1 + \cos(2\theta)\right) d\theta$$

$$= 25\left(\theta + \frac{1}{2}\sin(\theta)\right)\Big|_0^\pi$$

$$= 25\left(\left(\pi + 0\right) - 0\right)\Big) = 25\pi \checkmark$$



Is this really a circle?

Take $r = 10\cos\theta$. (Multiply both sides by r to get:)

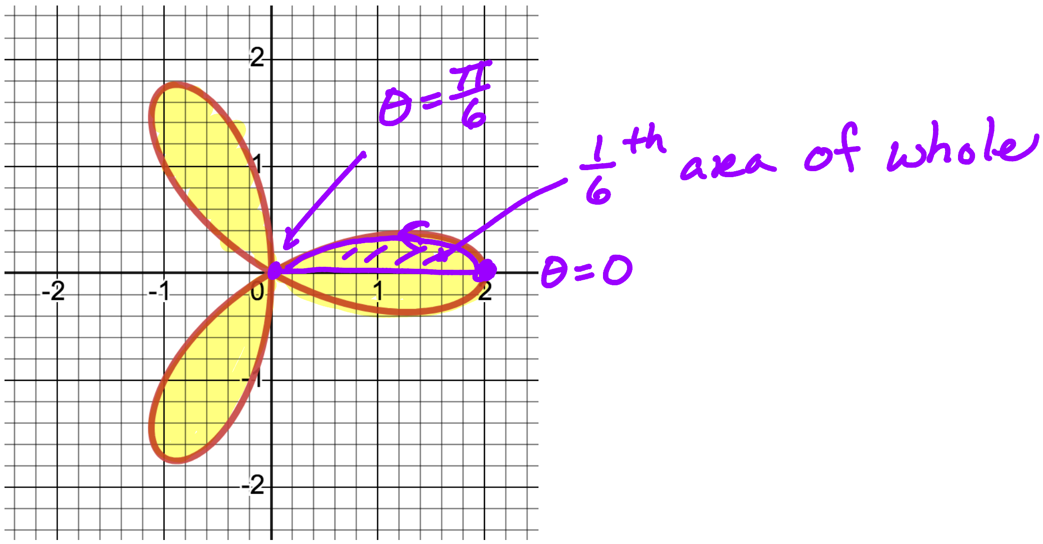
$r^2 = 10r\cos\theta$. (Use $x^2+y^2=r^2$ and $x = r\cos\theta$ to get:)

$x^2+y^2 = 10x$ (Use completing the square:

$x^2 - 10x + 25 + y^2 = 0 + 25$

$(x-5)^2 + y^2 = 25$ ✓ Yes! It's a circle!

(3) Let R be the region enclosed by the polar curve $r = 2\cos(3\theta)$. Shade the region R, then Set up and evaluate the integral to find the area of R.



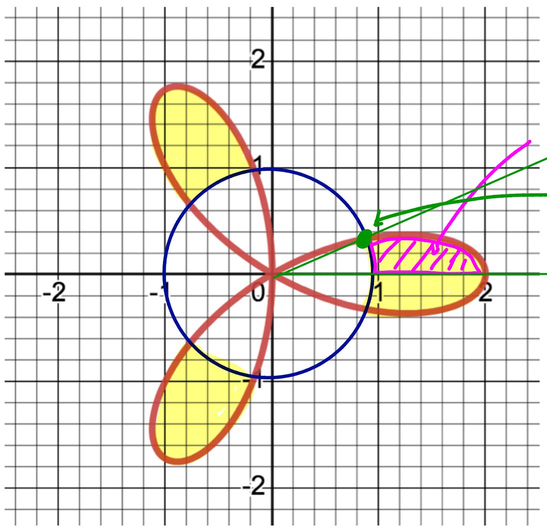
$$A = 6 \cdot \int_0^{\pi/6} \frac{1}{2}\left(2\cos(3\theta)\right)^2 d\theta$$

$$= 12 \int_0^{\pi/6} \cos^2(3\theta)\,d\theta = 6\int_0^{\pi/6} \left(1 + \cos(6\theta)\right) d\theta$$

$$= 6\left(\theta + \frac{1}{6}\sin(6\theta)\right)\Big|_0^{\pi/6}$$

$$= 6\left(\left(\frac{\pi}{6} + 0\right) - 0\right) = \pi$$

(4) Set up the integral to compute the area of the region inside $r = 2\cos(3\theta)$ and outside $r = 1$.



$\theta = \pi/9$

$\tilde{A}$. So $6\tilde{A} = A$

Find the point of intersection: $2\cos(3\theta) = 1$, $\cos(3\theta) = \frac{1}{2}$ So $3\theta = \frac{\pi}{3}$ or $\theta = \frac{\pi}{9}$

$\theta = 0$

$$A = 6\int_0^{\pi/9}\left[\tfrac{1}{2}(2\cos(3\theta))^2 - \tfrac{1}{2}(1)^2\right]d\theta = 3\int_0^{\pi/9}\left(4\cos^2(3\theta) - 1\right)d\theta = 3\int_0^{\pi/9}\left(2\left(1+\cos(6\theta)\right) - 1\right)d\theta$$

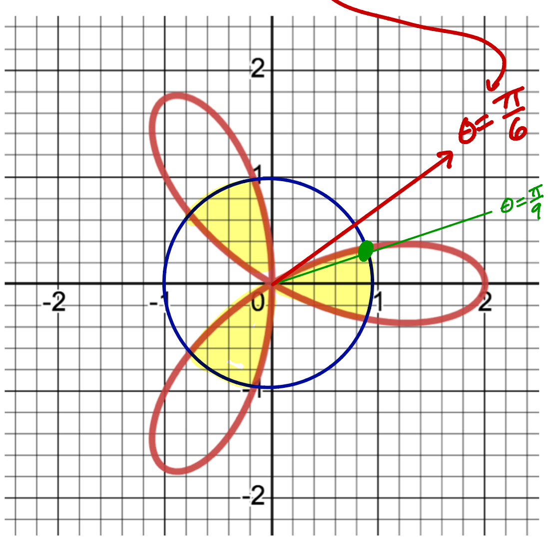
$$= 3\int_0^{\pi/9}\left(1 + 2\cos(6\theta)\right)d\theta = 3\left(\theta + \tfrac{1}{3}\sin(6\theta)\right)\Big|_0^{\pi/9}$$

$$= 3\left[\left(\frac{\pi}{9} + \frac{1}{3}\sin\left(\frac{6\pi}{9}\right)\right) + \underbrace{\left(0 + \tfrac{1}{3}\sin(0)\right)}_{0}\right] = \frac{\pi}{3} + \sin\left(\frac{2\pi}{3}\right) = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2} \approx 1.91$$

(5) Set up the integral to compute the area of the region inside *both* $r = 2\cos(3\theta)$ and $r = 1$.

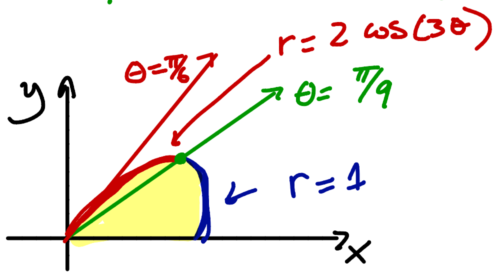
We know this from #3



$\theta = \frac{\pi}{6}$

$\theta = \frac{\pi}{9}$

We know this from #4 above. If we look closely, we see that r is bounded by different functions on either side of $\theta = \frac{\pi}{9}$. Here is my cartoon:



$$A = 6\left[\int_0^{\pi/9}\tfrac{1}{2}(1)^2 d\theta + \int_{\pi/9}^{\pi/6}\tfrac{1}{2}(2\cos(3\theta))^2 d\theta\right] = 6\left[\frac{\pi}{18} + \int_{\pi/9}^{\pi/6} 2\cos^2(3\theta)\,d\theta\right] = \frac{\pi}{3} + 6\int_{\pi/9}^{\pi/6}\left(1 + \cos(6\theta)\right)d\theta$$

$$= \frac{\pi}{3} + 6\left(\theta + \frac{1}{6}\sin(6\theta)\right)\Big|_{\pi/9}^{\pi/6} = \frac{\pi}{3} + 6\left[\left(\frac{\pi}{6} + 0\right) - \left(\frac{\pi}{9} + \frac{1}{6}\sin\left(\frac{2\pi}{3}\right)\right)\right]$$

$$= \frac{\pi}{3} + 6\left(\frac{\pi}{18} - \frac{\sqrt{3}}{12}\right) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \approx 1.22$$